Competitive Pricing in Multiplex Networks*

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Abstract

We study personalized pricing by two competing firms, each supplying a product associated with a distinct layer of a multiplex network. Products are substitutes, and consumers benefit from their neighbors' consumption within each layer. In equilibrium, firms offer deeper discounts to consumers with higher multiplex centrality on the rival layer. Stronger alignment of consumers' centralities across networks intensifies competition and lowers both firms' profits under sufficiently weak substitution. Expanding one network can reduce consumer surplus and social welfare despite lowering both firms' prices, because consumption diversion can destroy external benefits on the rival network that outweigh the gains from additional links.

Keywords: price competition; multiplex networks; consumption externality.

JEL codes: D85, L13, L11, D62

1 Introduction

Consumers often value a product more when their friends use it more intensively. These local network externalities need not operate through the same relationships across different products: a consumer may be central in the social network associated with one product but peripheral in the network associated with another.[1] Online gaming provides a natural illustration. A player may interact with different groups of

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[1]Zignani et al. (2016) find weak correlations between users' degree-based centrality rankings across most pairs of social media platforms. Similarly, studying users with accounts on both Twitter and Instagram, Lee and Lim (2016) find that their friendship networks often differ across the two platforms.

friends in different games, while spending more time in one game leaves less time for another.[2] Similar considerations arise when users divide their attention across social media platforms, such as Twitter and Instagram, where they may interact with different contacts.[3]

These product-specific relationships give competing firms different sources of demand among the same consumers. To compete for these consumers, firms can use targeted coupons and promotional discounts to tailor the prices individuals pay, accounting for how consumers' positions across competing networks shape their value to each firm. Consider a stylized version of the gaming example in which two multiplayer services charge for time played. Each firm can offer a player a personalized hourly rate, and the player's total payment on that service equals the quoted rate times the hours she plays. The firms' pricing decisions then depend on how each player's activity generates spillovers through her contacts on the two services. This paper studies competitive price discrimination in this multiplex network setting and addresses three questions. How do consumers' positions across these networks shape the discounts a firm offers? Does competition intensify when the same consumers are central in both networks? And does expanding one network necessarily benefit consumers?

We develop a two-stage model in which two firms simultaneously choose personalized prices, after which consumers decide their consumption of both products. Each product is associated with a distinct layer of a multiplex network connecting the same consumers. Consumption generates positive externalities among neighbors within each layer, while each consumer's activities are substitutes across layers. Intrinsic valuations and marginal costs are common across consumers and products, so differences in prices and consumption can be attributed to the network structure. Under a restriction on network strength and substitutability, we derive interior equilibrium prices, consumption, and profits in closed form.

Our first result shows how a consumer's position in one product-specific network translates into a discount on the competing product. Without cross-product substitution, the incentive to stimulate consumption spillovers and extract network benefits offset, yielding a uniform monopoly price. With substitution, equilibrium prices equal this uniform benchmark minus a personalized discount proportional to the consumer's multiplex centrality on the rival layer. The rival network creates both competitive pressure and an opportunity to attract demand beyond the targeted consumer. In particular, a discount

[2]For empirical evidence on peer influence within and across games, see Kim and Rao (2022), who study users' repeated visits to two social network games.

[3]In a field experiment, Aridor (2025) finds that restricting access to Instagram or YouTube leads participants to spend more time on other social applications.

draws her activity away from the rival product and consequently reduces the externalities she generates for her contacts, encouraging their consumption of the firm's own product. These discounts elicit reciprocal price responses that propagate across both networks. The resulting multiplex measure augments the rival network's standalone centrality with this cross-layer pricing feedback, so the firm's own network also shapes its equilibrium discounts.

We next isolate the role of cross-network centrality alignment. For two given product-specific networks, we hold consumers' positions in network A fixed and reassign them to positions in network B. This preserves each network's topology and distribution of standalone centralities while varying their alignment across the two networks. Stronger alignment means that consumers who are central in one network also tend to be central in the other. When the two products are sufficiently weak substitutes, stronger alignment lowers both firms' average prices and profits by intensifying competition. Misalignment in cross-layer network positions acts as a source of horizontal differentiation that softens competition even when each network's standalone characteristics are held fixed.

Our third result shows that network expansion can reduce consumer surplus and social welfare even when every equilibrium price falls. Adding links or strengthening network effects for one product creates benefits within its network, but can draw consumption away from the rival product. Consumers who reduce their use of that product generate fewer benefits for their contacts, even though their relationships remain intact. These losses can outweigh the gains from stronger connections and lower prices. Without cross-product substitution, this diversion channel disappears, and network expansion weakly increases both welfare measures.

Two extensions clarify the roles of competition and substitution. When two firms merge, they coordinate their price schemes and the optimal pricing scheme is uniform pricing across consumers and products, which raises price of each consumer and joint profit. When substitution is sufficiently strong, consumers consume only one product and there may exist multiple equilibria.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model. Section 4 characterizes equilibrium pricing and examines how network structure affects competition and welfare. Section 5 considers two extensions. Section 6 concludes. All proofs are collected in the appendix.

2 Related literature

Pricing network externalities. Our paper contributes to the literature on pricing with local network effects.[4] This literature examines how consumers' network positions and the information available about them shape firms' pricing decision. In the monopoly case, pricing balances the incentive to subsidize consumers for the spillovers they generate against the incentive to charge for the network benefits they receive (Candogan et al., 2012; Bloch and Quérou, 2013; Fainmesser and Galeotti, 2016). Greater influence therefore favors a discount, whereas greater exposure to others' consumption favors a higher price. These effects can offset: in the homogeneous linear-quadratic benchmark with undirected networks and constant marginal cost, uniform pricing is optimal.

Competition adds the incentive to attract consumption from rival products, giving network position a further role in discriminatory pricing (Chen et al., 2018a; Fainmesser and Galeotti, 2020; Chen et al., 2022).[5] Most closely related, Chen et al. (2018a) study personalized pricing for substitutable products on a common network. We retain their two-stage pricing game but allow product-specific relationship networks, nesting their benchmark when the effective networks coincide.[6] This extension allows us to isolate the effect of cross-network position alignment on competition while holding each network's structure fixed. It also shows that the consumer-welfare gains from network expansion in the common-network benchmark need not extend to product-specific networks. Tavasoli et al. (2023) also study pricing across distinct networks, using a stochastic sales-diffusion model to examine how interlayer correlation affects the steady states. They numerically optimize one firm's prices holding rival prices fixed; we instead characterize a simultaneous personalized-pricing equilibrium with utility-maximizing consumers.

Multi-activity and multiplex network games. The multi-activity structure of consumer demand connects our analysis to a broader literature on multi-activity network games. Building on the linear-quadratic

[4]Early work on platform pricing and competition considers global or cross-group network effects, including uniform pricing within each market side (Rochet and Tirole, 2003; Armstrong, 2006) as well as discriminatory pricing (Jullien, 2011; Liu and Serfes, 2013). A central consideration is that attracting a user generates not only direct revenue but also benefits for other users, whose participation and demand affect the platform's profitability. These benefits can also shape differentiation: Zhang and Sarvary (2015) show that local network effects in user-generated content can differentiate otherwise identical platforms.

[5]The broader literature on oligopolistic price discrimination examines how targeting consumers with different intrinsic preferences affects surplus extraction and competition (e.g., Thisse and Vives, 1988; Shaffer and Zhang, 1995; Corts, 1998; Shaffer and Zhang, 2002; Rhodes and Zhou, 2024). Our baseline holds intrinsic valuations constant across consumers to isolate the pricing implications of network position.

[6]A different extension is Li et al. (2026b), who introduce positive cross-platform consumption spillovers through interoperability while retaining a common relationship network.

framework of Ballester et al. (2006), Chen et al. (2018b) and Kor and Zhou (2023) examine how within-activity network spillovers interact with substitution or complementarity across an individual's activities. The pricing model of Chen et al. (2018a) also belongs to this literature: its consumer subgame treats consumption of competing products as interdependent activities on a common network.

By extending to product-specific relationship networks, our paper also belongs to the growing research agenda on multiplex networks, where the same agents can interact with different neighbors across layers. The broader network-science literature examines their structure and dynamics (Kivelä et al., 2014; Cozzo et al., 2018). Related economic research studies information diffusion (Chandrasekhar et al., 2026), network formation (Joshi et al., 2020; Billand et al., 2023; Joshi et al., 2025), exchange economies (Li et al., 2026a), cooperation (Cheng et al., 2021), and public goods provision (Walsh, 2019). Closest to our setting, Zenou and Zhou (2026) study effort allocation across heterogeneous network layers, whereas we examine personalized price competition between firms that anticipate consumers' cross-layer responses. They characterize equilibrium activities in terms of layer-specific weighted Katz–Bonacich centralities, with endogenous weights determined jointly across layers. For our pricing analysis, we derive a block-matrix representation that expresses consumer demand as an explicit affine function of both firms' prices. This representation makes consumers' price responses directly available for solving the pricing game, and allows us to characterize how network interactions and price competition jointly determine equilibrium prices and consumption.

3 Model

3.1 Consumers and product-specific networks

There are two firms, A and B, each supplying a product, and a common set of consumers $N = \{1, \ldots, n\}$. Each product $s \in \{A, B\}$ is associated with an undirected weighted relationship network $G^s = (g^s_{ij})_{i,j \in N}$, which governs the consumption externalities among its users.[7] We assume that G^s is symmetric, nonnegative and has a zero diagonal. The two networks are exogenously given and need not coincide: the same consumers may interact with the same sets of peers in different relationship patterns when consuming different products or services.

[7] We use G^s to denote both the network and its adjacency matrix, relying on context to distinguish the two.

Consumer i chooses consumption level $x_i^s \geq 0$ of each product s. Her intrinsic marginal value is $a_i^s > 0$, and firm s charges her a personalized per-unit price p_i^s. The utility generated by consumption (x_i^A, x_i^B) is

$$\begin{aligned} u_i = {} & a_i^A x_i^A + a_i^B x_i^B - \frac{1}{2}(x_i^A)^2 - \frac{1}{2}(x_i^B)^2 - \beta x_i^A x_i^B \\ & + \delta_A x_i^A \sum_{j \in N} g_{ij}^A x_j^A + \delta_B x_i^B \sum_{j \in N} g_{ij}^B x_j^B - p_i^A x_i^A - p_i^B x_i^B, \end{aligned} \tag{1}$$

The parameter $\delta_s > 0$ measures the strength of local positive network spillovers. The parameter $\beta \in [0, 1)$ measures substitutability between the two products. In a platform application, it can capture the substitution effect arising from interoperability. For example, when the same content is accessible through interoperable interfaces, consuming it on one platform can partially replace consumption on the other. This interpretation is consistent with Chen (2026), who models greater interoperability as an increase in substitutability across platforms.

The substitution term has a direct attention interpretation. The private cost of the two activities can be decomposed as

$$\frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B = \frac{1-\beta}{2}\left[(x_i^A)^2 + (x_i^B)^2\right] + \frac{\beta}{2}(x_i^A + x_i^B)^2.$$

The first term is product-specific activity cost, and the second is a convex cost of total attention. Unlike a hard time constraint, this specification leaves total consumption endogenous.

The price term represents a monetary payment per unit of consumption, as in usage-based pricing for paid digital services, online games, communication services, or creator tools.

Our utility specification builds on Chen et al. (2018a), who study competition between two substitute products with local consumption complementarities on a common consumer network. Setting $G^A = G^B = G$ and $\delta_A = \delta_B = \delta$ recovers their utility specification. We allow both the consumer network and the strength of network effects to vary across products. This extension allows us to examine how differences in consumers' network positions across products shape personalized pricing and competition.

3.2 Timing

Firm s charges consumer i a per-unit price p_i^s and incurs a constant marginal cost c_i^s with $c_i^s < a_i^s$. Let $\mathbf{x}^s, \mathbf{a}^s$, and $\mathbf{p}^s$ collect the corresponding consumer-level variables. Given prices and suppose $\mathbf{x}^s(\mathbf{p}^A, \mathbf{p}^B) \geq 0$,[8] its profit is

$$\pi^s(\mathbf{p}^A, \mathbf{p}^B) = \sum_{i \in N} (p_i^s - c_i^s) x_i^s(\mathbf{p}^A, \mathbf{p}^B) = (\mathbf{p}^s - \mathbf{c}^s)^\top \mathbf{x}^s(\mathbf{p}^A, \mathbf{p}^B).$$

We define a game Γ with two stages:

1. In stage 1, the two firms simultaneously choose personalized price vectors $\mathbf{p}^s \in \mathbb{R}^n$, $s \in \{A, B\}$. A negative price p_i^s represents a per-unit subsidy to consumer i.

2. In stage 2, after observing both price vectors, consumers simultaneously choose consumption of both products .

We study the pure strategy subgame-perfect equilibrium by backward induction. For convenience, we refer to the second-stage game as the consumption game.

4 Main Results

4.1 Equilibrium consumption in the second stage

We solve game Γ through backward induction. Consider the consumption game first. Given a price profile $(\mathbf{p}^A, \mathbf{p}^B)$, the first-order conditions of an interior consumption equilibrium are

$$\begin{cases} x_i^A + \beta x_i^B = a_i^A - p_i^A + \delta_A \sum_{j \in N} g_{ij}^A x_j^A, \\ x_i^B + \beta x_i^A = a_i^B - p_i^B + \delta_B \sum_{j \in N} g_{ij}^B x_j^B. \end{cases} \tag{2}$$

The sign pattern separates two interactions. Within a network, neighbors' activities are *strategic complements*. Across networks, a consumer's own activities are *substitutes*. If $g_{ij}^A > 0$, an increase in j's consumption of product A raises i's consumption of A and can thereby reduce i's consumption of B. The Hessian of i's payoff with respect to her own choices is negative definite under Assumption 1. Hence, the

[8] In section 4, we show that under some regularity conditions, the interior solution is guaranteed and unique.

first-order conditions characterize her interior best response. Stacking the first-order conditions in (2) across consumers gives

$$Q\begin{pmatrix}\mathbf{x}^A\\ \mathbf{x}^B\end{pmatrix}=\begin{pmatrix}\mathbf{a}^A-\mathbf{p}^A\\ \mathbf{a}^B-\mathbf{p}^B\end{pmatrix}\quad\text{where}\quad Q=\begin{pmatrix}I-\delta_A G^A & \beta I\\ \beta I & I-\delta_B G^B\end{pmatrix} \tag{3}$$

We denote the diagonal blocks by $Q^s = I-\delta_s G^s$ for $s\in\{A,B\}$.

Assumption 1. $\beta+\max\{\delta_A\rho(G^A),\delta_B\rho(G^B)\}<1$.

Under Assumption 1, we can define within-layer multiplier $K^s=(Q^s)^{-1}=\sum_{\ell=0}^{\infty}(\delta_s G^s)^\ell\geq 0$ to aggregate direct and indirect effects transmitted along within-layer network walks.[9] Its row sums, collected in $\mathbf{b}^s=(I-\delta_s G^s)^{-1}\mathbf{1}$, give consumers' Katz–Bonacich centralities (Katz, 1953; Bonacich, 1987). The following proposition characterizes the resulting consumption equilibrium.

Proposition 1. *Suppose Assumption 1 holds. For every pair of price vectors, the unconstrained consumption game has a unique equilibrium,*[10]

$$\begin{pmatrix}\mathbf{x}^A\\ \mathbf{x}^B\end{pmatrix}=M\left[\begin{pmatrix}\mathbf{a}^A\\ \mathbf{a}^B\end{pmatrix}-\begin{pmatrix}\mathbf{p}^A\\ \mathbf{p}^B\end{pmatrix}\right],\qquad \textit{where}\qquad M=Q^{-1}=\begin{pmatrix}M^{AA} & M^{AB}\\ M^{BA} & M^{BB}\end{pmatrix}. \tag{4}$$

Furthermore, the diagonal blocks satisfy $M^{ss}\succ 0$ and $M^{ss}\geq 0$, whereas the off-diagonal blocks satisfy $M^{sr}\leq 0$ for all $s,r\in\{A,B\}$ with $s\neq r$.[11]

Proposition 1 directly implies

$$\begin{cases}\mathbf{x}^A=M^{AA}(\mathbf{a}^A-\mathbf{p}^A)+M^{AB}(\mathbf{a}^B-\mathbf{p}^B),\\ \mathbf{x}^B=M^{BA}(\mathbf{a}^A-\mathbf{p}^A)+M^{BB}(\mathbf{a}^B-\mathbf{p}^B).\end{cases}$$

When consumption vectors are positive, an increase in any consumer's price for one product weakly reduces every consumer's consumption of that product and weakly raises consumption of the rival product.

[9] Throughout, matrix inequalities are componentwise, with $>$ and $<$ strict in every component. The symbol 0 denotes the appropriate scalar, vector, or matrix depending on the context.

[10] By "unconstrained," we mean that the nonnegativity constraints on consumption are relaxed, allowing $x_i^s\in\mathbb{R}$ rather than $x_i^s\geq 0$. The equilibrium in (4) may therefore involve negative consumption for some price vectors. Whenever the solution in (4) is nonnegative, it is also the unique equilibrium of the consumption subgame.

[11] For symmetric matrices, let $\succ 0$ denote positive definiteness.

The consumer facing the price increase strictly reduces her consumption of that product, as the positive definiteness of M^{AA} and M^{BB} implies strictly positive diagonal entries.

Remark 1. Our consumption subgame is related to Section V.A of Zenou and Zhou (2026) which investigates multiplex game under linear-quadratic utility functions. Their sufficient spectral condition for equilibrium uniqueness is equivalent to our Assumption 1.[12] Zenou and Zhou (2026) express equilibrium activity on each layer in terms of a weighted Katz–Bonacich centrality:

$$\begin{cases} \mathbf{x}^{s*} = (I - \phi_s G^s)^{-1} \left(\mathbf{a}^s - \mathbf{p}^s - \dfrac{1}{\alpha^s} \boldsymbol{\mu}^* \right), \\ \boldsymbol{\mu}^* = c \displaystyle\sum_{s \in S} \mathbf{x}^{s*}. \end{cases}$$

Here, $(I - \phi_s G^s)^{-1}$ is constructed using only the network on layer s, whereas the weights depend on the endogenous vector $\boldsymbol{\mu}^*$. Its ith component is the marginal cost of individual i's total effort across all layers, and c is their effort-cost parameter. Thus, activities across layers are linked through the weights rather than through $(I - \phi_s G^s)^{-1}$ itself.

In our pricing game, a change in either firm's prices can alter consumption across layers and hence the marginal-cost vector $\boldsymbol{\mu}^*$. Deriving optimal prices from this representation therefore requires accounting for the adjustment of $\boldsymbol{\mu}^*$ to price changes. By contrast, Proposition 1 incorporates these adjustments directly into the block matrix M, expressing the entire consumption profile as an explicit affine function of price vectors. This representation provides a tractable demand system for characterizing personalized pricing and its comparative statics in multiplex networks.

4.2 Competitive personalized pricing

From this section onward, we impose homogeneous primitives across consumers and firms.[13]

$$a_i^A = a_i^B = a \text{ and } c_i^A = c_i^B = c \text{ for each } i \in N \tag{5}$$

Thus $\mathbf{a}^s = a\mathbf{1}$ and $\mathbf{c}^s = c\mathbf{1}$. This specialization removes valuation and cost heterogeneity from all subsequent comparisons, so cross-consumer and cross-firm differences in equilibrium prices and activities are generated

[12] The normalization gives $\delta_s = (1-\beta)\phi_s$, where ϕ_s is their within-layer spillover coefficient. Their restriction $\phi_s \rho(G^s) < 1$ for each layer is therefore equivalent to $\beta + \max_s\{\delta_s \rho(G^s)\} < 1$.

[13] This assumption of homogeneity also arises as the full symmetry assumption in Chen et al. (2018a).

only by $\delta_A G^A$ and $\delta_B G^B$. For convenience, we refer to $\delta_s G^s$ as an effective network for product s.

Assumption 2. $\beta + \max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\} < 1$ *where* $d^s_{\max} = \max_{i\in N} \sum_{j\in N} g^s_{ij} = \|G^s\|_\infty$.

By the standard spectral-radius bound for induced matrix norms, we have $\rho(G^s) \le \|G^s\|_\infty = d^s_{\max}$ where $\|G^s\|_\infty$ is the induced infinity norm. Thus, Assumption 2 is stronger than Assumption 1, which means the preceding results continue to hold.

To derive the equilibrium personalized prices, we introduce a $2n \times 2n$ weighted matrix that captures pricing feedback within and across the two network layers:

$$W = \begin{pmatrix} \delta_A G^A & \frac{\beta}{2} I \\ \frac{\beta}{2} I & \delta_B G^B \end{pmatrix}.$$

This block-matrix representation follows the supra-adjacency formulation of multilayer networks described by De Domenico et al. (2013); Kivelä et al. (2014).[14] Its nodes are consumer–layer pairs (i,s), where $i \in N$ and $s \in \{A,B\}$. Each consumer therefore appears once on each layer. Importantly, W is a *pricing-feedback* network, rather than the primitive consumer-interaction network. Within a common effective network, $\delta_A G^A = \delta_B G^B = \delta G$, our matrix Q becomes $Q = \widetilde{\Phi} \otimes I_n - I_2 \otimes \delta G$, where $\widetilde{\Phi} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$. This is the two-activity representation in Kor and Zhou (2023).

- *Within-layer blocks.* The diagonal blocks are $W_{AA} = \delta_A G^A$ and $W_{BB} = \delta_B G^B$. An edge between (i,s) and (j,s) has weight $\delta_s g^s_{ij}$, which equals the within-layer strategic complements: $\frac{\partial^2 u_i}{\partial x^s_i \partial x^s_j} = \delta_s g^s_{ij} > 0$ whenever $g^s_{ij} > 0$.

- *Cross-layer blocks.* The off-diagonal blocks are $W_{AB} = W_{BA} = \frac{\beta}{2} I_n$. They describe links between the two copies of the same consumer: (i,A) and (i,B) are connected with weight $\beta/2$. [15]

  These blocks are diagonal because cross-product substitution is individual-specific: $\frac{\partial^2 u_i}{\partial x^A_i \partial x^B_i} = -\beta$, and $\frac{\partial^2 u_i}{\partial x^A_i \partial x^B_j} = 0$ for $j \neq i$. Thus, there is no direct cross-layer link between distinct consumers.

Definition 1 (Multiplex Katz–Bonacich centrality)**.** Under Assumption 2, the multiplex Katz–Bonacich

[14] W is also an unfolding of the fourth-order adjacency tensor, obtained by grouping each consumer–layer pair into a single row or column index. For more discussion, see De Domenico et al. (2013).

[15] If firm A's best response to $\mathbf{p}^B$ induces strictly positive consumption of both products for every consumer, its pricing first-order condition gives $\mathbf{p}^A = \frac{a+c}{2}\mathbf{1} - \frac{\beta}{2} K^B (a\mathbf{1} - \mathbf{p}^B)$. Within this interior region, $\frac{\partial p^A_i}{\partial p^B_j} = \frac{\beta}{2}(K^B)_{ij}$. This derivative measures the local response of A's optimal price to consumer i to a marginal change in B's price to consumer j. The coefficient $\beta/2$ captures direct pricing feedback, and K^B captures its propagation through the rival network.

centrality associated with W is given by

$$\widehat{\mathbf{b}} = \begin{pmatrix} \widehat{\mathbf{b}}^A \\ \widehat{\mathbf{b}}^B \end{pmatrix} = (I_{2n} - W)^{-1}\mathbf{1} = \sum_{k=0}^{\infty} W^k \mathbf{1}$$

We refer to $\widehat{\mathbf{b}}$ as the multiplex Katz–Bonacich centrality to distinguish it from the usual Katz–Bonacich centrality defined on each layer, and retain the centrality of each pair rather than aggregating across layers.[16]

The first two powers of matrix W illustrate the mechanism. At $k = 1$,

$$W\mathbf{1}_{2n} = \begin{pmatrix} \delta_A G^A & \frac{\beta}{2} I \\ \frac{\beta}{2} I & \delta_B G^B \end{pmatrix} \mathbf{1}_{2n} = \begin{pmatrix} \delta_A G^A + \frac{\beta}{2} I \\ \delta_B G^B + \frac{\beta}{2} I \end{pmatrix} \mathbf{1}_n.$$

Each node is directly affected by its within-layer neighbors and by its own vertical cross-layer link. At $k = 2$,

$$W^2\mathbf{1}_{2n} = \begin{pmatrix} \delta_A^2 (G^A)^2 + \frac{\beta^2}{4} I & \frac{\beta}{2}\left(\delta_A G^A + \delta_B G^B\right) \\ \frac{\beta}{2}\left(\delta_A G^A + \delta_B G^B\right) & \delta_B^2 (G^B)^2 + \frac{\beta^2}{4} I \end{pmatrix} \mathbf{1}_{2n} = \begin{pmatrix} \delta_A^2 (G^A)^2 + \frac{\beta^2}{4} I + \frac{\beta}{2}\left(\delta_A G^A + \delta_B G^B\right) \\ \delta_B^2 (G^B)^2 + \frac{\beta^2}{4} I + \frac{\beta}{2}\left(\delta_A G^A + \delta_B G^B\right) \end{pmatrix} \mathbf{1}_n.$$

Two-step feedback therefore consists of: (*i*) two within-layer steps, captured by $\delta_s^2 (G^s)^2$; (*ii*) a vertical round trip by the same consumer, captured by $(\beta^2/4) I_n$; and (*iii*) mixed paths containing one within-layer step and one cross-layer step, captured by $\frac{\beta}{2}\left(\delta_s G^s + \delta_r G^r\right)$. The off-diagonal blocks of W^2 are generally no longer diagonal: once feedback crosses layers, it can propagate through the consumer network associated with the other firm. Higher powers of W consequently generate walks that alternate between the two layers arbitrarily often. Therefore the centrality vector $\widehat{\mathbf{b}}$ depends jointly on G^A and G^B, rather than on either network in isolation.[17]

The following theorem shows that, under Assumption 2, the price vectors are associated with this centrality measure and the equilibrium consumption is positive.

Theorem 1. *Suppose Assumption 2 holds. For $s, r \in \{A, B\}$ with $r \neq s$, the unique equilibrium outcome is*

[16] See also De Domenico et al. (2015, Supplementary Note 3, p. 34, Eq. (13)) for an early discussion on the node–layer Katz centrality of multilayer networks.

[17] As a consequence, $\widehat{b}_i^r$ and $\widehat{b}_i^s$ are correlated. To see this, by definition $\widehat{\mathbf{b}} = (I_{2n} - W)^{-1}\mathbf{1}$. Multiplying both sides by $I_{2n} - W$ and taking the second block row of this matrix equation gives $-\frac{\beta}{2} I_n \widehat{\mathbf{b}}^r + (I_n - \delta_s G^s)\widehat{\mathbf{b}}^s = \mathbf{1}$. Rearranging this equation yields $\widehat{\mathbf{b}}^s = K^s\mathbf{1} + \frac{\beta}{2} K^s \widehat{\mathbf{b}}^r = \mathbf{b}^{\mathbf{s}} + \frac{\beta}{2} K^s \widehat{\mathbf{b}}^r$.

interior, characterized as follows.

(i) The equilibrium prices set by firm s are $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{\mathbf{b}}^r$.

(ii) The equilibrium consumption of product s is $\mathbf{x}^{s*} = \frac{a-c}{2}M^{ss}\left(\mathbf{1} - \frac{\beta}{2}\widehat{\mathbf{b}}^r\right)$.

(iii) The equilibrium profit of firm s is $\pi^{s*} = \frac{(a-c)^2}{4}\left(\mathbf{1} - \frac{\beta}{2}\widehat{\mathbf{b}}^r\right)^\top M^{ss}\left(\mathbf{1} - \frac{\beta}{2}\widehat{\mathbf{b}}^r\right)$.

The following corollary recovers two benchmark cases: no substitutability and coinciding effective networks.

Corollary 1. *Suppose Assumption 2 holds:*

(i) If $\beta = 0$*, then* $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1}$ *for each* $s \in \{A, B\}$.

(ii) If $G^A = G^B = G$ *and* $\delta_A = \delta_B = \delta$*, then* $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1} - \frac{(a-c)\beta}{2}\left[(2-\beta)I - 2\delta G\right]^{-1}\mathbf{1}$ *for each* $s \in \{A, B\}$.

Thus, the common-network formula on monopoly pricing (Bloch and Quérou, 2013; Candogan et al., 2012), and oligopoly pricing (Chen et al., 2018a) are nested as two special cases of Theorem 1.[18] The additional contribution of our framework is to allow the two firms to operate on distinct networks, in which case equilibrium discounts are governed by multiplex centrality rather than centrality in either layer alone.

Theorem 1 shows that firm A offers larger discounts to consumers who are more central on the rival layer. Specifically, consumer i's discount relative to the benchmark price $\frac{a+c}{2}$ is $h_i^A = \frac{(a-c)\beta}{4}\widehat{b}_i^B$. For $\beta > 0$, a higher $\widehat{b}_i^B$ therefore implies a lower price from firm A.

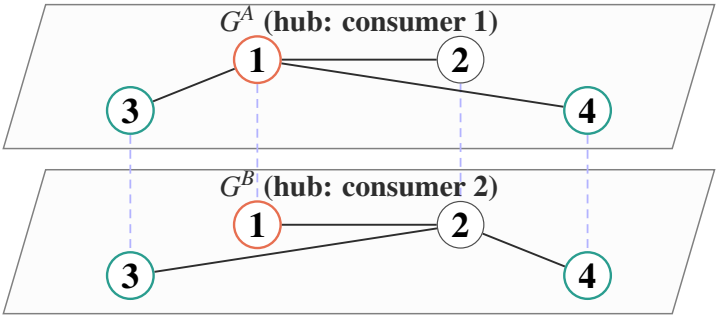
This pricing pattern reflects two opposing incentives within a firm's own consumer network. A central consumer generates positive spillovers that stimulate other users' consumption, giving the firm an incentive to subsidize her activity. However, she also receives network benefits from other users, which raise her willingness to pay and allow the firm to charge a higher price. When $\beta = 0$, these incentives offset and the firms charge the uniform price $\frac{a+c}{2}$, consistent with the monopoly-pricing benchmark of Bloch and Quérou (2013).

Competition introduces a different role for network position. Firm A sets the prices of its own product, taking the rival's prices as given. The network externalities associated with product B make consumption of

[18] In the common network case, $\delta_A G^A = \delta_B G^B = \delta G$, inverting the block matrix Q using the Schur complement yields $M^{AA} = M^{BB} = \frac{M^+ + M^-}{2}, M^{AB} = M^{BA} = \frac{M^+ - M^-}{2}$, where $M^+ := [(1+\beta)I_n - \delta G]^{-1}$ and $M^- := [(1-\beta)I_n - \delta G]^{-1}$ are defined in Chen et al. (2018b) and Chen et al. (2018a) for their analysis of multiple activities on a common network.

that product more attractive and reduce consumers' willingness to pay for consumption of product A. These benefits therefore create competitive pressure that A must accommodate through its own prices. Consumers whose positions on the rival layer expose A to stronger competitive pressure receive larger discounts. The resulting price differentiation is thus driven by the need to compete for activity that could otherwise take place on the rival network.

Finally, the relevant measure $\widehat{b}_i^B$ incorporates pricing feedback across both networks. Firm A's own network can therefore still affect its equilibrium prices by changing B's pricing incentives, which feed back into the discount offered by A.



Consumer i	b_i^A	b_i^B	$\widehat{b}_i^B$	p_i^{A*}
1	1.55	1.23	1.74	4.61
2	1.23	1.55	2.10	4.32
3	1.23	1.23	1.64	4.68
4	1.23	1.23	1.64	4.68

Figure 1: Multiplex centrality and personalized pricing. Consumers 1, 3, and 4 have identical standalone centrality on network B. Consumer 1's higher multiplex centrality associated with B nevertheless leads to a larger discount and a lower price from firm A. Light blue dashed lines connect the same consumer across layers.

Example 1. *Figure 1 illustrates the pricing implications of Theorem 1. Consider four consumers positioned on two star networks. Let* $a = 10, c = 2, \delta_A = \delta_B = 0.15, \beta = 0.4$. *As* $d_{\max}^A = d_{\max}^B = 3$, *these parameters satisfy assumption 2:* $\beta + \max_{s\in\{A,B\}}\{\delta_s d_{\max}^s\} = 0.4 + 0.15 \times 3 = 0.85 < 1$.

The first comparison is between consumer 1 and consumer 2. Although consumer 2 is peripheral in G^A, *like consumers 3 and 4, she is the hub of* G^B *and has the highest multiplex centrality associated with that network. Consequently, consumer 2 receives the largest discount from firm A and pays the lowest price ,* $p_2^{A*} \approx 4.32$.

The second comparison is between consumer 1 and consumers 3 and 4. When G^B *is considered in isolation, these consumers occupy equivalent peripheral positions and have identical centrality,* $b_1^B = b_3^B = b_4^B \approx 1.23$. *Their positions across the two networks nevertheless differ: consumer 1 is the hub of* G^A, *whereas consumers 3 and 4 are peripheral on both networks. This difference is reflected in their multiplex*

centralities associated with G^B: $\widehat{b}_1^B \approx 1.74$, compared with $\widehat{b}_3^B = \widehat{b}_4^B \approx 1.64$. Consumer 1 therefore receives a larger discount and pays $p_1^{A*} \approx 4.61$, compared with $p_3^{A*} = p_4^{A*} \approx 4.68$.

To summarize, neither own-network degree nor standalone rival-network centrality alone determines personalized prices.

4.3 Centrality Alignment and Competition

Consumers may occupy similar or different positions across networks: a consumer who is central on one network may also be central on the other, or may instead occupy a peripheral position. We examine how multiplexity affects competition by considering how consumers' network positions differ across the two networks. To isolate this effect, we fix two arbitrary networks G^A and G^B, together with their network-effect parameters δ_A and δ_B. We hold consumers' positions on G^A fixed and vary their assignment to positions on G^B. This comparison preserves each network's structure and changes only how consumers' positions correspond across networks. The two networks need not be isomorphic.

Definition 2 (Permutation matrix)**.** An $n \times n$ matrix P is a permutation matrix if its entries belong to $\{0,1\}$ and each row and each column contains exactly one entry equal to 1. Let $\mathscr{P}_n := \{P \in \{0,1\}^{n\times n} : P\mathbf{1} = \mathbf{1}, P^\top \mathbf{1} = \mathbf{1}\}$ denote the set of all such matrices. Every $P \in \mathscr{P}_n$ satisfies $P^{-1} = P^\top$.

For each $P \in \mathscr{P}_n$, consider the network pair G^A and $G^B(P) = PG^BP^\top$. Here, $P_{ij} = 1$ means that consumer i on $G^B(P)$ occupies position j of G^B. Consumer identities and positions on G^A remain fixed. Changing P therefore preserves each layer's network structure, number of links, and degree distribution, while allowing a given consumer to occupy different positions across the two layers. The centrality vector on G^A remains $\mathbf{b}^A$, whereas that on $G^B(P)$ becomes $\mathbf{b}^B(P) = (I - \delta_B PG^BP^\top)^{-1}\mathbf{1} = P\mathbf{b}^B$.[19]

Consider G^B in Figure 1, where consumer 2 is connected to consumers 1, 3, and 4. Suppose consumers 1 and 2 exchange their positions, while consumers 3 and 4 remain in their original positions. The

[19] Since $P^{-1} = P^\top$ and $P^\top\mathbf{1} = \mathbf{1}$, we have $(I - \delta_B PG^BP^\top)^{-1} = P(I - \delta_B G^B)^{-1}P^\top$. Multiplying both sides by $\mathbf{1}$ gives $\mathbf{b}^B(P) = P\mathbf{b}^B$.

corresponding permutation matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad G^B(P) = PG^BP^\top = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

After the permutation, consumer 1 occupies the central position of the star network. The centrality vector becomes $\mathbf{b}^B(P) = P\mathbf{b}^B = (b_2^B, b_1^B, b_3^B, b_4^B)^\top$. Thus, consumers 1 and 2 exchange their centralities in G^B. Their positions in G^A remain unchanged, so consumer 1 is now central in both networks.

Definition 3 (Cross-platform centrality alignment)**.** For $P \in \mathscr{P}_n$, define

$$S(P) = (\mathbf{b}^A)^\top \mathbf{b}^B(P) = \sum_{i \in N} b_i^A (P\mathbf{b}^B)_i = \frac{1}{2}\left(\|\mathbf{b}^A\|_2^2 + \|\mathbf{b}^B\|_2^2 - \|\mathbf{b}^A - P\mathbf{b}^B\|_2^2\right).$$

as the cross-platform centrality alignment index.

A larger $S(P)$ indicates stronger positive alignment of consumers' centralities across platforms: consumers who are central on one platform tend also to be central on the other.[20] Thus, holding $G^A, G^B, \delta_A, \delta_B$ fixed, stronger centrality alignment is equivalent to a smaller sum of squared differences between consumers' centralities across platforms.[21]

We hold $G^A, G^B, \delta_A, \delta_B$ fixed and suppress dependence on these parameters in the notation. For each firm $s \in \{A, B\}$, let $\overline{p}^{s*}(P, \beta) = \frac{1}{n}\mathbf{1}^\top \mathbf{p}^{s*}(P, \beta)$ denote its equilibrium average price under the network pair G^A and $G^B(P) = PG^BP^\top$. Comparisons across permutations are made at the same value of β.

Theorem 2. *Suppose that* $\max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\} < 1$. *Consider two permutation matrices* $P_0, P_1 \in \mathscr{P}_n$ *satisfying* $S(P_1) > S(P_0)$. *There exists* $\overline{\beta} \in \left(0, 1 - \max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\}\right)$ *such that, for every* $\beta \in (0, \overline{\beta})$,

$$\begin{cases} \overline{p}^{s*}(P_1, \beta) < \overline{p}^{s*}(P_0, \beta), \\ \pi^{s*}(P_1, \beta) < \pi^{s*}(P_0, \beta), \end{cases} \qquad s \in \{A, B\}.$$

[20] This index measures the alignment of consumers' centralities; it does not directly measure the overlap of links across platforms. If either network is regular, its centrality vector is constant and $S(P)$ is invariant to P.

[21] By the rearrangement inequality (Hardy et al., 1952), $S(P)$ is maximized by matching positions in the same order of standalone centrality across the two reference networks.

Theorem 2 shows that competition depends on how consumers' network positions correspond across layers. When substitutability is sufficiently small and positive, stronger centrality alignment lowers both firms' average prices and profits. Each firm offers larger discounts to consumers who are more central on the rival network. Stronger alignment directs these discounts toward consumers who also have higher baseline consumption on the firm's own network, making the discounts more costly. Differences in consumers' positions across networks can therefore soften competition without changing either network's overall connectivity or the substitutability of the two products. This result provides a testable prediction: stronger cross-network centrality alignment is associated with lower average prices and profits when substitution is sufficiently weak.

4.4 Comparative statics

We now examine how equilibrium outcomes vary with valuations, costs, substitutability, network effects, and network density. Throughout, we maintain the assumption of homogeneous primitives.

The following proposition gives the comparative statics for equilibrium prices and consumption.

Proposition 2. *Suppose Assumption 2 holds. For $s, r \in \{A, B\}$ with $r \neq s$, the following statements hold.*

(i) Valuation and marginal cost: $\dfrac{\partial \mathbf{p}^{s*}}{\partial a} > 0, \quad \dfrac{\partial \mathbf{p}^{s*}}{\partial c} > 0, \quad \dfrac{\partial \mathbf{x}^{s*}}{\partial a} = -\dfrac{\partial \mathbf{x}^{s*}}{\partial c} > 0.$

(ii) Substitutability: $\dfrac{\partial \mathbf{p}^{s*}}{\partial \beta} < 0.$

(iii) Product-specific network effects: $\dfrac{\partial \mathbf{p}^{s*}}{\partial \delta_s} \leq 0, \quad \dfrac{\partial \mathbf{p}^{r*}}{\partial \delta_s} \leq 0, \quad \dfrac{\partial \mathbf{x}^{s*}}{\partial \delta_s} \geq 0.$

(iv) Denser networks: Suppose $G^{s\prime} \geq G^s$. Holding G^r and all other primitives fixed,

$$\mathbf{p}^{s*}(G^{s\prime}, G^r) \leq \mathbf{p}^{s*}(G^s, G^r), \qquad \mathbf{p}^{r*}(G^{s\prime}, G^r) \leq \mathbf{p}^{r*}(G^s, G^r), \qquad \mathbf{x}^{s*}(G^{s\prime}, G^r) \geq \mathbf{x}^{s*}(G^s, G^r).$$

A higher valuation a raises both prices and consumption, whereas a higher marginal cost c raises prices but reduces consumption. An increase in substitutability β intensifies competition between the two firms, leading each firm to lower its price to every consumer.

A stronger network effect for product s weakly lowers both firms' prices and weakly increases consumption of product s. Similarly, a denser network G^s weakly lowers both firms' prices and weakly increases

consumption of product s. If $\delta_s > 0$ and $G^{s\prime} \neq G^s$, aggregate consumption of product s strictly increases. The effect on consumption of the rival product r, however, is ambiguous.

Proposition 3 (Comparative statics on profits)**.** *Suppose Assumption 2 holds. For $s, r \in \{A, B\}$ with $r \neq s$, the following statements hold.*[22]

(i) The equilibrium profits satisfy $\dfrac{\partial \pi^{s*}}{\partial a} = -\dfrac{\partial \pi^{s*}}{\partial c} > 0$, $\dfrac{\partial \pi^{s*}}{\partial \beta} < 0$ *and* $\dfrac{\partial \pi^{r*}}{\partial \delta_s} \leq 0$.

(ii) Suppose $G^{s\prime} \geq G^s$ and hold G^r and all other primitives fixed, $\pi^{r*}(G^{s\prime}, G^r) \leq \pi^{r*}(G^s, G^r)$. Moreover, for sufficiently small β, $\pi^{s*}(G^{s\prime}, G^r) \geq \pi^{s*}(G^s, G^r)$.

Proposition 3 distinguishes between changes in the underlying gains from trade and changes in the competitive environment. A higher common valuation increases consumers' baseline willingness to pay, whereas a higher common marginal cost reduces the surplus available from supplying products. Their equal and opposite effects on equilibrium profits reflect the role of the common net valuation, $a - c$: increasing valuation and reducing marginal cost by the same amount have equivalent effects on firm profitability.

Greater substitutability strengthens the crowding-out relationship between consumers' consumption of the two products. Through the term $-\beta x_i^A x_i^B$, consumption of one product reduces the marginal value of consumption of the other, and this effect becomes stronger as β increases. Each firm consequently faces a tighter constraint on its ability to monetize consumer activity. Personalized pricing allows firms to tailor their offers to individual consumers, but it does not remove the competitive constraint imposed by the rival's product. The decline in both firms' profits thus reflects the stronger interaction between competing uses and the resulting pressure on profitability.

Stronger network effects for product s generate a negative competitive spillover onto firm r. An increase in δ_s raises the value that consumers obtain from their neighbors' consumption of product s, strengthening the incentives for activity within that network. When the products are substitutes, stronger spillovers on network s divert demand from product r toward product s at the initial price profile. The associated decline in firm s's equilibrium prices reinforces this diversion. Firm r's own price reductions can more than offset these effects. Nevertheless, at firm r's pricing optimum, the profit from sales generated by a marginal price cut exactly offsets the loss of margin on existing sales, so that firm r's own price response has no first-order

[22] At $\beta = 0$ or $\delta_s = 0$, derivatives are understood from the admissible side. "Denser" here means a componentwise increase in the weighted adjacency matrix, not a rewiring that preserves only the total number of links.

effect on its profit. The equilibrium profit effect is therefore $\frac{\partial \pi^{r*}}{\partial \delta_s} = -\beta(\mathbf{x}^{r*})^\top K^s G^s \mathbf{x}^{s*} + \beta(\mathbf{x}^{r*})^\top K^s \frac{\partial \mathbf{p}^{s*}}{\partial \delta_s} \leq 0$, where the first term captures the profit loss from demand diversion at fixed prices, and the second captures the additional loss from firm s's equilibrium price reductions. The case $\beta = 0$ isolates this transmission mechanism: without substitution between products, the two firms' demand systems are separable, and an improvement in one product's network effects leaves the rival's profit unchanged.

The effect on firm s's own profit reflects a balance between the direct benefit of a stronger network and the consequences of equilibrium pricing adjustments. Holding the rival's prices fixed, stronger network effects improve the opportunities to generate profits from consumer activity. In equilibrium, however, the rival also adjusts its personalized prices, and competitive price concessions can limit the improving firm's ability to capture the additional network value. This consideration is particularly relevant under personalized pricing because a consumer's activity generates both direct revenue and indirect benefits through its influence on other consumers. Stronger network effects can therefore increase the value of attracting activity and also strengthen incentives to compete for it. The direct benefit need not dominate the resulting competitive pressure, explaining why own profit can move in either direction. When $\beta = 0$, this cross-firm competitive feedback disappears, and stronger network effects strictly increase own profit whenever the network is nonempty.

An expansion of G^s operates through a related mechanism by adding or strengthening connections within firm s's network. With positive network effects and substitutability, such an expansion reduces the rival's profit, but its effect on own profit remains ambiguous because the additional network benefits interact with both firms' pricing responses. Thus, strengthening a firm's network can weaken its competitor without necessarily improving its own profitability.

4.5 Consumer surplus and social welfare

In this section, we examine how consumer surplus (CS) and social welfare (SW) vary with model parameters and network structure. We define $CS = \sum_{i \in N} u_i(\mathbf{x}^A, \mathbf{x}^B; \mathbf{p}^A, \mathbf{p}^B)$ and $SW = CS + \pi^A + \pi^B$.

Lemma 1 (Welfare expressions)**.** *Suppose Assumption 2 holds. Equilibrium aggregate consumer surplus*

and social welfare are given by

$$\begin{cases} CS^* = \dfrac{1}{2}\|\mathbf{x}^{A*}\|_2^2 + \dfrac{1}{2}\|\mathbf{x}^{B*}\|_2^2 + \beta(\mathbf{x}^{A*})^\top \mathbf{x}^{B*}, \\ SW^* = (a-c)\mathbf{1}^\top(\mathbf{x}^{A*} + \mathbf{x}^{B*}) - \dfrac{1}{2}\|\mathbf{x}^{A*}\|_2^2 - \dfrac{1}{2}\|\mathbf{x}^{B*}\|_2^2 - \beta(\mathbf{x}^{A*})^\top \mathbf{x}^{B*} \\ \qquad\quad + \delta_A(\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*} + \delta_B(\mathbf{x}^{B*})^\top G^B \mathbf{x}^{B*}. \end{cases} \tag{6}$$

Valuation and marginal cost. Based on the expressions of CS^* and SW^* in Lemma 1, we can directly calculate that $\frac{\partial CS^*}{\partial a} = -\frac{\partial CS^*}{\partial c} > 0$ and $\frac{\partial SW^*}{\partial a} = -\frac{\partial SW^*}{\partial c} > 0$. Thus, a higher valuation a increases both consumer surplus and social welfare, whereas a higher marginal cost c reduces both.

Cross-product substitutability. A higher β lowers equilibrium prices but also increases the cost of consuming both products. Its welfare effects therefore depend on the resulting consumption responses and the associated changes in network benefits.

Observation 1 (Welfare effects of substitutability)**.** *Under Assumption 2, the effects of β on equilibrium consumer surplus and social welfare are ambiguous: both $\frac{\partial CS^*}{\partial \beta}$ and $\frac{\partial SW^*}{\partial \beta}$ can be positive or negative, even when both effective networks are regular.*

To demonstrate the ambiguity, suppose that the effective networks of the two products have the same row sum $d \in (0,1)$: $\delta_A G^A \mathbf{1} = \delta_B G^B \mathbf{1} = d\mathbf{1}$. Let $\eta = 1-d$. Each consumer chooses $x_i^{A*} = x_i^{B*} = q = \frac{(a-c)\eta}{(\eta+\beta)(2\eta-\beta)}$ where $0 \le \beta < \eta$. The corresponding welfare levels are

$$CS^* = n(1+\beta)q^2, \qquad SW^* = n\left[2(a-c)q - (2\eta - 1 + \beta)q^2\right].$$

Fix $\eta = 1/4$, $a = 2$, and $c = 1$.[23] Table 1 reports equilibrium prices, consumption, and welfare at three values of β.

The arrows indicate the direction of change as β increases at the reported value. At $\beta = 0$, derivatives are evaluated from the right.

The table illustrates the interaction between substitution and price competition. When β is close to zero, stronger substitution reduces equilibrium consumption, consumer surplus, and social welfare. As

[23] The particular values of a and c do not affect the directions of the comparative statics reported below. For any $a > c$, equilibrium consumption is proportional to $a-c$, while consumer surplus and social welfare are proportional to $(a-c)^2$. Equilibrium prices are affine in c and $a-c$, so their responses to β also have the same signs.

Table 1: Equilibrium prices, consumption, and welfare

β	p_i^{s*}	q	CS^*	SW^*
0	1.50 ($\downarrow$)	2.00 ($\downarrow$)	$4.00n$ ($\downarrow$)	$6.00n$ ($\downarrow$)
$1/6$	1.25 ($\downarrow$)	1.80 ($\uparrow$)	$3.78n$ ($\uparrow$)	$4.68n$ ($\uparrow$)
$1/5$	1.17 ($\downarrow$)	1.85 ($\uparrow$)	$4.12n$ ($\uparrow$)	$4.73n$ ($\uparrow$)

β increases, the price response becomes stronger and eventually outweighs the substitution effect on consumption. Both consumer surplus and social welfare increase as β rises from $1/6$ to $1/5$. At $\beta = 1/5$, consumer surplus exceeds its level at $\beta = 0$, whereas social welfare remains below its level at $\beta = 0$.

The ambiguity in the consumer-surplus response is consistent with part (*i*) of Corollary 3 in Chen et al. (2018a). They study price competition on a common network and also find that the price reductions induced by greater substitutability need not increase consumer surplus or social welfare.

Stronger network effects and additional links. We distinguish between stronger network effects and network expansion. For the former, we vary δ_A, holding G^A, G^B, δ_B, a, c, and β fixed. For the latter, we hold all scalar parameters fixed and compare G^A with a denser network $G^{A\prime}$, keeping G^B unchanged. Throughout, all parameter values and networks being compared satisfy Assumption 2. When writing $CS^*(G^A, G^B)$ and $SW^*(G^A, G^B)$, we suppress the scalar parameters that are held fixed.

Proposition 4. *Suppose Assumption 2 holds, the comparative statics about CS and SW are as follows.*

(i) If $\beta = 0$, then, for each $s \in \{A, B\}$, we have $\frac{\partial CS^*}{\partial \delta_s} \geq 0$ and $\frac{\partial SW^*}{\partial \delta_s} \geq 0$. Moreover, if $G^{s\prime} \geq G^s$, then we have $CS^*(G^{s\prime}, G^r) \geq CS^*(G^s, G^r)$ and $SW^*(G^{s\prime}, G^r) \geq SW^*(G^s, G^r)$.

(ii) Fix a nonnegative matrix D. There exists $\varepsilon_D > 0$ such that, whenever $\max_{s\in\{A,B\}} \|\delta_s G^s - D\|_\infty < \varepsilon_D$, we have $\frac{\partial CS^*}{\partial \delta_s} \geq 0$ and $\frac{\partial SW^*}{\partial \delta_s} \geq 0$, for each $s \in \{A, B\}$. For a finite expansion from D to $D' \geq D$. There exists $\varepsilon_{D,D'} > 0$ such that, whenever $G^{A\prime} \geq G^A$, $G^{B\prime} \geq G^B$, and $\max_{s\in\{A,B\}} \{\|\delta_s G^s - D\|_\infty, \|\delta_s G^{s\prime} - D'\|_\infty\} < \varepsilon_{D,D'}$, we have $CS^*(G^{A\prime}, G^{B\prime}) \geq CS^*(G^A, G^B)$ and $SW^*(G^{A\prime}, G^{B\prime}) \geq SW^*(G^A, G^B)$.

(iii) For general asymmetric effective networks, the signs of $\frac{\partial CS^*}{\partial \delta_s}$ and $\frac{\partial SW^*}{\partial \delta_s}$ are not determined. Likewise, expanding one network can increase or decrease both welfare measures.

Proposition 4 highlights why strengthening one network need not improve welfare. Adding links to one network or strengthening its network effects increases the benefits generated by consumption on that network. When $\beta = 0$, the two products are independent, so these changes do not affect consumption on

the rival network and thus increases both consumer surplus and social welfare, as established in part (i) of Proposition 4.

When $\beta > 0$, however, strengthening one network can draw consumption away from the rival network and thus the welfare effect is ambiguous. Part (ii) of Proposition 4 demonstrates that if the difference between the two effective networks is sufficiently small, expanding one network will increase CS and SW. This finding generalizes a network-monotonicity result in Corollary 3 of Chen et al. (2018a) who finds that expanding a common network will increase consumer surplus.

With positive substitution and different effective networks, however, additional links in one network is insufficient to establish a welfare improvement: the loss of external benefits on the rival network can outweigh the gains from stronger network effects or additional links. To make this mechanism explicit, rearrange the price-derivative identity in the proof of Lemma 2:

$$\begin{aligned}\frac{\partial CS^*}{\partial \delta_A} = &\underbrace{-\sum_{s\in\{A,B\}} (\mathbf{x}^{s*})^\top \frac{\partial \mathbf{p}^{s*}}{\partial \delta_A}}_{\text{gains from lower prices}} + \underbrace{(\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*}}_{\text{direct gains from stronger network effects}} \\ &+ \underbrace{\delta_A (\mathbf{x}^{A*})^\top G^A \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A}}_{\text{additional spillovers on } A} + \underbrace{\delta_B (\mathbf{x}^{B*})^\top G^B \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A}}_{\text{changes in spillovers on } B}.\end{aligned} \tag{7}$$

This is an exact equilibrium decomposition. All price and consumption derivatives incorporate both firms' pricing adjustments and consumers' resulting equilibrium responses.

The first three terms are nonnegative. Increasing δ_A weakly lowers both firms' prices, directly strengthens the benefits generated by existing relationships in G^A, and increases consumption of product A. The latter response creates additional benefits for other consumers in that network.

The fourth term, however, can be negative. Although δ_B and G^B remain fixed, consumer i's marginal network benefit for product B depends on her neighbors' consumption through $\delta_B \sum_{j\in N} g^B_{ij} x^{B*}_j$. If those neighbors reduce their consumption of product B, this benefit falls. The final term in (7) aggregates these changes, weighting consumption responses by the external benefits they generate for other consumers. When the resulting loss is sufficiently large, consumer surplus decreases despite stronger network effects for product A and lower prices for both products. The same economic trade-off arises in a finite network comparison.

Example 2. *A four-consumer example illustrates that either adding a link or strengthening existing network effects can reduce consumer surplus and social welfare. Let* $N = \{1,2,3,4\}$*, and* $a = 1$*,* $c = 0$*,* $\beta = \frac{1}{40}$*,* $\delta_A = \frac{1}{3}$*, and* $\delta_B = \frac{97}{100}$*. Initially, the edge sets are* $G^A = \{\{1,2\},\{3,4\}\}$ *and* $G^B = \{\{1,2\}\}$*. Let* G^A *and* G^B *denote the two original networks.* G^A *initially connects consumers* 1 *and* 2*, and separately connects consumers* 3 *and* 4*.* G^B *connects only consumers* 1 *and* 2*.*

First, hold all parameters fixed and add a link $\{2,3\}$ *to* G^A*, giving* $G^{A\prime} = G^A \cup \{\{2,3\}\}$*. Let* $G^{A\prime}$ *denote the resulting network. The maximum degree in A increases from* 1 *to* 2*, whereas that in B remains* 1*. Both the initial and final networks satisfy Assumption 2 as* $\beta + \max\{\delta_A d^{A\prime}_{\max}, \delta_B d^B_{\max}\} = \frac{1}{40} + \max\left\{\frac{2}{3}, \frac{97}{100}\right\} = \frac{199}{200} < 1$*.*

The equilibrium consumption levels, individual utility, and aggregate welfare measures are reported below:

	Before the link addition			*After the link addition*		
Consumer i	x_i^{A*}	x_i^{B*}	u_i^*	x_i^{A*}	x_i^{B*}	u_i^*
1	0.4430	16.7437	140.4595	0.5810	16.6610	139.2050
2	0.4430	16.7437	140.4595	0.8457	16.6598	139.4838
3	0.7411	0.4910	0.4043	1.0584	0.4904	0.6934
4	0.7411	0.4910	0.4043	0.8469	0.4908	0.4895
CS^*		281.73			279.87	
SW^*		299.49			297.77	

Every consumer increases consumption of product A and reduces consumption of product B. Both firms also lower all personalized prices. Nevertheless, aggregate consumer surplus and social welfare both decrease.

Consumers 3 *and* 4 *benefit from the expansion of A's network, but consumers* 1 *and* 2 *lose surplus. The latter reduce their consumption of product B, where their relationship generates much stronger externalities. Their combined surplus falls by approximately* 2.23*, exceeding the approximately* 0.37 *gain enjoyed by consumers* 3 *and* 4*.*

Second, hold the resulting networks $(G^{A\prime}, G^B)$ *fixed and consider an increase in* δ_A *from* $1/3$*, keeping all other parameters unchanged. This strengthens every existing relationship in A without adding any links. Differentiating the equilibrium welfare expressions with respect to* δ_A *gives* $\left.\frac{\partial CS^*}{\partial \delta_A}\right|_{\delta_A = 1/3} \approx -3.20$ *and* $\left.\frac{\partial SW^*}{\partial \delta_A}\right|_{\delta_A = 1/3} \approx -1.11$*. Hence, both welfare measures strictly decrease for every sufficiently small positive*

increase in δ_A.

These two comparisons illustrate how strengthening one network can reduce welfare through cross-product consumption diversion. Adding $\{2,3\}$ creates a new relationship, whereas increasing δ_A strengthens all existing relationships in product A's network. In both cases, consumption of product A increases and both firms lower their prices, but consumption of product B declines. Because the relationship between consumers 1 and 2 generates strong externalities in G^B, even a small reduction in their consumption can destroy substantial network benefits. In this example, these losses outweigh the gains from the stronger network and higher consumption of product A.

Without cross-product substitution, this diversion channel is absent, and network expansion increases welfare. With substitution, its welfare effect also depends on the loss of network benefits caused by lower consumption of the rival's product. Network asymmetry therefore matters: additional links and lower prices do not by themselves guarantee higher consumer surplus or social welfare.

5 Extensions

5.1 Merger

We next examine how a merger changes pricing across the multiplex network. Under separate ownership, each firm chooses its prices without accounting for the resulting change in the rival's profit. A merger allows a single firm to coordinate both price vectors and internalize these effects. The merged firm retains both products and chooses personalized prices to maximize their combined profit,

$$\Pi(\mathbf{p}^A, \mathbf{p}^B) = \sum_{s \in \{A,B\}} (\mathbf{p}^s - c\mathbf{1})^\top \mathbf{x}^s(\mathbf{p}^A, \mathbf{p}^B). \tag{8}$$

We use a tilde to denote the merged firm's optimal prices, the induced equilibrium consumption, and its maximized total profit.

Proposition 5. *Suppose Assumption 2 holds. The merged firm's unique optimal price vectors are* $\tilde{\mathbf{p}}^A = \tilde{\mathbf{p}}^B = \frac{a+c}{2}\mathbf{1}$. *The corresponding total profit is* $\tilde{\Pi} = \frac{(a-c)^2}{4}\mathbf{1}_{2n}^\top Q^{-1}\mathbf{1}_{2n}$. *If* $\beta > 0$, *the merger strictly increases every consumer's price for both products:* $\tilde{\mathbf{p}}^s > \mathbf{p}^{s*}$ *for each* $s \in \{A,B\}$, *and strictly increases total profit:* $\tilde{\Pi} > \pi^{A*} + \pi^{B*}$.

Proposition 5 shows that common ownership eliminates the competitive discounts generated by separate ownership. When firm A lowers its price, it attracts consumption from B but does not account for the resulting loss of B's profit. The merged firm incorporates this loss into its pricing decision and therefore removes the incentive for such discounting. The incentive to subsidize consumers for the network benefits they generate exactly offsets the incentive to charge more for the network benefits they receive. Consequently, the merged firm charges the uniform price $\frac{a+c}{2}$, even when consumers occupy different positions across the two networks. This finding is consistent with the network-independent monopoly prices in undirected networks established by Candogan et al. (2012) and Bloch and Quérou (2013). Chen et al. (2018a) show that joint-profit-maximizing prices are also independent of network structure when two substitutable products share a common network. Our result extends this conclusion to products associated with distinct networks. Network structure and substitutability continue to affect consumption and profit, but do not affect the optimal prices under merger.

5.2 Large Substitutability

We now allow $\beta > 1$ and examine consumers' product choices at given prices. Strong substitution makes simultaneous consumption costly. We show that participating consumersconsume only one product and characterize how prices and network benefits determine their incentives to switch.

Proposition 6. *If $\beta > 1$, no consumer consumes a positive amount of both products at stage 2.*

For a consumer i choosing product A, the no-switch condition is $p_i^A - p_i^B \leq \delta_A \sum_j g_{ij}^A x_j^A - \delta_B \sum_j g_{ij}^B x_j^B$.

When $\beta > 1$, network effects can support different patterns of consumer coordination, which in turn affect firms'pricing incentives. Consequently, the entire two-stage game may have multiple equilibria, with different prices and market allocations, complicating both equilibrium pricing and network comparative statics. The following example establishes this possibility.

Example 3. *Consider two consumers and suppose $G^A = G^B = \{12\}$, $a = 1$, $c = 0$, $\delta_A = \delta_B = \frac{3}{4}$, and $\beta > 1$. The two-stage game admits two pure-strategy subgame-perfect equilibria with the following outcomes:*[24]

$$\begin{cases} \textit{Equilibrium 1:} \quad (\mathbf{p}^{A*}, \mathbf{p}^{B*}; \mathbf{x}^{A*}, \mathbf{x}^{B*}) = \left((\frac{1}{2}, \frac{1}{2})^\top, \mathbf{0}; (2,2)^\top, \mathbf{0} \right), \\ \textit{Equilibrium 2:} \quad (\mathbf{p}^{A*}, \mathbf{p}^{B*}; \mathbf{x}^{A*}, \mathbf{x}^{B*}) = \left(\mathbf{0}, (\frac{1}{2}, \frac{1}{2})^\top; \mathbf{0}, (2,2)^\top \right). \end{cases} \tag{9}$$

[24] In Appendix, we verify that each outcome in (9) is indeed an equilibrium.

The corresponding profit pairs are $(2,0)$ *and* $(0,2)$. *These equilibria remain valid even if firms are allowed to offer negative prices.*

The example shows how consumer coordination can sustain different equilibrium prices and market allocations under identical primitives. Although the inactive firm charges zero, a consumer would lose the network benefit generated by the other consumer if she switched alone. Attracting consumers despite this coordination requires a subsidy sufficiently large to make entry unprofitable. Either firm can therefore serve the entire market and charge a positive price in a subgame-perfect equilibrium. This multiplicity makes network comparative statics more demanding: changes in network structure can affect both pricing incentives and the consumer equilibrium selected after a price change.

6 Conclusion

This paper shows how product-specific relationship networks shape personalized pricing and the welfare consequences of competition. Equilibrium discounts reflect consumers' multiplex centrality in the rival network. Even when the firms possess identical network structures and centrality distributions, stronger alignment of consumers' centralities lowers both firms' average prices and profits under sufficiently weak positive substitution. Differences in consumers' network positions can therefore soften competition while preserving each network's overall connectivity. Network expansion introduces a further trade-off: additional connections generate benefits for one product but can draw consumption away from the other, weakening the external benefits available there. Both consumer surplus and social welfare can consequently fall despite lower prices. The extensions show that common ownership eliminates competitive discounts, while sufficiently strong substitution leads participating consumers to consume only one product and can support multiple equilibrium price profiles. These findings suggest that assessing firm competition requires attention to how consumers' network positions correspond across products, and that evaluating network expansion requires accounting for the external benefits lost when consumption shifts between them.

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A Proofs

Proof of Proposition 1. Equation (3) expresses the consumer first-order conditions as a linear system. We first show that its coefficient matrix Q is positive definite and hence invertible. For any vectors z^A, z^B, we have $2(z^A)^\top z^B \geq -\|z^A\|^2 - \|z^B\|^2$ where $\|\cdot\|$ denotes the Euclidean norm: $\|z\| = (z^\top z)^{1/2} = (\sum_{i=1}^n z_i^2)^{1/2}$ for $z \in \mathbb{R}^n$. Recall G^s is the adjacency matrix of an undirected network, so it is real-symmetric and elementwise nonnegative. By the Perron–Frobenius theorem for nonnegative matrices, its spectral radius coincides with its largest eigenvalue: $\rho(G^s) = \lambda_{\max}(G^s)$. By the Rayleigh–Ritz variational principle for Hermitian matrices (Horn and Johnson, 2012, Theorem 4.2.2), of which real symmetric matrices are a special case, we obtain

$\frac{z^\top G^s z}{z^\top z} \le \lambda_{\max}(G^s) = \rho(G^s)$ for any nonzero vector z, i.e., $z^\top G^s z \le \rho(G^s) z^\top z$. Then for $z = \begin{pmatrix} z^A \\ z^B \end{pmatrix} \neq 0$.

$$
\begin{aligned}
z^\top Q z =& \begin{pmatrix} z^A \\ z^B \end{pmatrix}^\top \begin{pmatrix} Q^A & \beta I \\ \beta I & Q^B \end{pmatrix} \begin{pmatrix} z^A \\ z^B \end{pmatrix} \\
=& (z^A)^\top Q^A z^A + (z^B)^\top Q^B z^B + 2\beta\, (z^A)^\top z^B \\
=& (z^A)^\top (I - \delta_A G^A) z^A + (z^B)^\top (I - \delta_B G^B) z^B + 2\beta\, (z^A)^\top z^B \\
\ge& \|z^A\|^2 - \delta_A \rho(G^A) \|z^A\|^2 + \|z^B\|^2 - \delta_B \rho(G^B) \|z^B\|^2 + \beta(-\|z^A\|^2 - \|z^B\|^2) \\
=& \left[1 - \delta_A \rho(G^A) - \beta\right] \|z^A\|^2 + \left[1 - \delta_B \rho(G^B) - \beta\right] \|z^B\|^2 > 0.
\end{aligned}
$$

The last inequality follows from Assumption 1. Thus Q is positive definite and hence invertible, establishing uniqueness of the solution to the first-order conditions and yielding (4). For $s, r \in \{A, B\}$ with $r \neq s$, the Schur-complement formula gives

$$
\begin{cases}
M^{ss} = \left[Q^s - \beta^2 (Q^r)^{-1}\right]^{-1}, \\
M^{sr} = -\beta M^{ss} (Q^r)^{-1}.
\end{cases}
\tag{A.1}
$$

First consider $\beta = 0$. In this case, $Q = \mathrm{diag}(Q^A, Q^B), M = \mathrm{diag}(K^A, K^B)$. Under Assumption 1, for each $s \in \{A, B\}$, $K^s = (I - \delta_s G^s)^{-1} = \sum_{\ell=0}^{\infty} (\delta_s G^s)^\ell \ge 0$, and $K^s \succ 0$. Hence $M^{ss} \ge 0$, $M^{ss} \succ 0$, and $M^{sr} = 0$ for $s \neq r$, establishing the claims when $\beta = 0$. For the remainder of this proof, suppose $\beta > 0$.

As G^A is component-wise nonnegative, so is $\delta_A G^A$. Moreover, Assumption 1 implies that $\rho(\delta_A G^A) = \delta_A \rho(G^A) \le \max\{\delta_A \rho(G^A), \delta_B \rho(G^B)\} < 1 - \beta < 1$. Therefore, $Q^A = I - \delta_A G^A$ is a nonsingular M-matrix. The same reasoning applies to $Q^B = I - \delta_B G^B$. Nonsingular M-matrices possess elementwise nonnegative inverses, so $(Q^A)^{-1} \ge 0$ and $(Q^B)^{-1} \ge 0$.

Now consider the Schur complement $S_A = Q^A - \beta^2 (Q^B)^{-1}$. We rewrite $S_A = I - \left(\delta_A G^A + \beta^2 (Q^B)^{-1}\right)$. Define $L = \delta_A G^A + \beta^2 (Q^B)^{-1}$. As $\delta_A G^A \ge 0$ and $(Q^B)^{-1} \ge 0$, L is elementwise nonnegative. Because $\delta_A G^A$ and $\beta^2 (Q^B)^{-1}$ are both symmetric and elementwise nonnegative, the Rayleigh–Ritz variational principle yields $\rho(L) = \lambda_{\max}(L) \le \lambda_{\max}(\delta_A G^A) + \lambda_{\max}\left(\beta^2 (Q^B)^{-1}\right) = \delta_A \rho(G^A) + \beta^2 \rho\left((Q^B)^{-1}\right)$. Under Assumption 1, $\delta_A \rho(G^A) < 1 - \beta$ and $\delta_B \rho(G^B) < 1 - \beta$. As Q^B is symmetric, $\rho\left((Q^B)^{-1}\right) = \frac{1}{1 - \delta_B \rho(G^B)}$.

Using $1-\delta_B\rho(G^B)>\beta$, we obtain

$$\rho(L)\le\delta_A\rho(G^A)+\frac{\beta^2}{1-\delta_B\rho(G^B)}<(1-\beta)+\frac{\beta^2}{\beta}=1.$$

Therefore S_A satisfies the definition of a nonsingular M-matrix. By symmetric argument, $S_B=Q^B-\beta^2(Q^A)^{-1}$ is also a nonsingular M-matrix. Thus $M^{AA}=S_A^{-1}\ge 0$ and $M^{BB}=S_B^{-1}\ge 0$.

As $\beta\ge 0$, $M^{AA}\ge 0$ and $(Q^B)^{-1}\ge 0$, the identity $M^{AB}=-\beta M^{AA}(Q^B)^{-1}$ implies that $M^{AB}\le 0$. The same logic yields $M^{BA}\le 0$. As $Q\succ 0$, its inverse $M=Q^{-1}\succ 0$. All principal submatrices of a positive definite matrix are positive definite, so the diagonal blocks satisfy $M^{AA}\succ 0$ and $M^{BB}\succ 0$.

□

Proof of Theorem 1. Under the homogeneous primitives assumption (5), Proposition 1 gives $\mathbf{x}^A=M^{AA}(a\mathbf{1}-\mathbf{p}^A)+M^{AB}(a\mathbf{1}-\mathbf{p}^B)$. As $\frac{\partial\mathbf{x}^A}{\partial\mathbf{p}^A}=-M^{AA}$ and M^{AA} is symmetric, the first-order condition for the profit of firm A is $\mathbf{x}^A-M^{AA}(\mathbf{p}^A-c\mathbf{1})=0$. The Hessian is $-2M^{AA}\prec 0$, so this condition is necessary and sufficient for firm A's profit maximization within the region of strictly positive consumption of both products. Substitution gives $2M^{AA}(\mathbf{p}^A-c\mathbf{1})+M^{AB}(\mathbf{p}^B-c\mathbf{1})=(a-c)(M^{AA}+M^{AB})\mathbf{1}$. Symmetrically, we obtain $2M^{BB}(\mathbf{p}^B-c\mathbf{1})+M^{BA}(\mathbf{p}^A-c\mathbf{1})=(a-c)(M^{BB}+M^{BA})\mathbf{1}$. Using the block expressions for M from the proof of Proposition 1, we can rewrite the interior pricing first-order conditions as:

$$\mathbf{p}^A=\frac{a+c}{2}\mathbf{1}-\frac{\beta}{2}K^B(a\mathbf{1}-\mathbf{p}^B), \tag{A.2}$$

$$\mathbf{p}^B=\frac{a+c}{2}\mathbf{1}-\frac{\beta}{2}K^A(a\mathbf{1}-\mathbf{p}^A). \tag{A.3}$$

Stacking the two firms' first-order conditions gives

$$\underbrace{\begin{pmatrix}2M^{AA} & M^{AB}\\ M^{BA} & 2M^{BB}\end{pmatrix}}_{M+\mathrm{diag}(M^{AA},M^{BB})}\begin{pmatrix}\mathbf{p}^A\\ \mathbf{p}^B\end{pmatrix}=\begin{pmatrix}(a+c)M^{AA}\mathbf{1}+aM^{AB}\mathbf{1}\\ aM^{BA}\mathbf{1}+(a+c)M^{BB}\mathbf{1}\end{pmatrix}.$$

The coefficient matrix is $M+\mathrm{diag}(M^{AA},M^{BB})$. Since Assumption 2 implies Assumption 1, $M=Q^{-1}$, M^{AA}, and M^{BB} are positive definite. Hence $M+\mathrm{diag}(M^{AA},M^{BB})\succ 0$, so the pricing first-order conditions admit a unique solution. We next derive this candidate and then establish global optimality and uniqueness,

allowing for zero consumption.

To solve explicitly for this candidate, define the discount of each firm $\mathbf{h}^s := \frac{a+c}{2}\mathbf{1} - \mathbf{p}^s$. As $a\mathbf{1} - \mathbf{p}^s = \frac{a-c}{2}\mathbf{1} + \mathbf{h}^s$, the interior pricing first-order conditions (A.2) and (A.3) become $\mathbf{h}^A = \frac{(a-c)\beta}{4}K^B\mathbf{1} + \frac{\beta}{2}K^B\mathbf{h}^B, \mathbf{h}^B = \frac{(a-c)\beta}{4}K^A\mathbf{1} + \frac{\beta}{2}K^A\mathbf{h}^A$. Premultiplying the first equation by Q^B and the second by Q^A, and using $Q^sK^s = I$, gives $Q^B\mathbf{h}^A - \frac{\beta}{2}\mathbf{h}^B = \frac{(a-c)\beta}{4}\mathbf{1}, Q^A\mathbf{h}^B - \frac{\beta}{2}\mathbf{h}^A = \frac{(a-c)\beta}{4}\mathbf{1}$. Ordering the unknowns as (h^B, h^A), we obtain

$$\underbrace{\begin{pmatrix} Q^A & -\frac{\beta}{2}I \\ -\frac{\beta}{2}I & Q^B \end{pmatrix}}_{I_{2n}-W} \begin{pmatrix} \mathbf{h}^B \\ \mathbf{h}^A \end{pmatrix} = \frac{(a-c)\beta}{4}\mathbf{1}.$$

Note that $\|W\|_\infty = \max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\} + \frac{\beta}{2} < 1 - \beta + \frac{\beta}{2} = 1 - \frac{\beta}{2} \leq 1$. Then $\rho(W) \leq ||W||_\infty < 1$ implies $I_{2n} - W$ is positive definite and thus invertible.

Then this system has the unique solution

$$\begin{pmatrix} \mathbf{h}^B \\ \mathbf{h}^A \end{pmatrix} = \frac{(a-c)\beta}{4}(I_{2n} - W)^{-1}\mathbf{1} = \frac{(a-c)\beta}{4}\begin{pmatrix} \widehat{\mathbf{b}}^A \\ \widehat{\mathbf{b}}^B \end{pmatrix}.$$

This proves the equilibrium price formulas. In particular, each firm's discount is proportional to the centrality on the *rival* layer.

Recall that the firm first-order conditions give $\mathbf{x}^{A*} = M^{AA}(\mathbf{p}^{A*} - c\mathbf{1})$ and $\mathbf{x}^{B*} = M^{BB}(\mathbf{p}^{B*} - c\mathbf{1})$. Using the pricing formula just derived proves the expressions of equilibrium consumption. The formulas for M^{AA} and M^{BB} are given in the proof of Proposition 1.

The profit part uses both the price and the consumption expression just derived. Note that $\mathbf{p}^{A*} - c\mathbf{1} = \frac{a-c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{\mathbf{b}}^B = (a-c)\left(\frac{1}{2}\mathbf{1} - \frac{\beta}{4}\widehat{\mathbf{b}}^B\right)$ and $x^{A*} = M^{AA}\left(\frac{a-c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{b}^B\right) = (a-c)M^{AA}\left(\frac{1}{2}\mathbf{1} - \frac{\beta}{4}\widehat{\mathbf{b}}^B\right)$. Substituting these expressions into $\pi^{A*} = (\mathbf{p}^{A*} - c\mathbf{1})^\top x^{A*}$, we obtain the expression of profit for firm A . The analysis for firm B is symmetric thus omitted.

We first verify the interiority of the candidate. If $\beta = 0$, the pricing and consumption formulas give $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1}$ and $\mathbf{x}^{s*} = \frac{a-c}{2}K^s\mathbf{1} > \mathbf{0}$ for each $s \in \{A, B\}$. Now suppose $\beta > 0$. As $W \geq 0$ and $W\mathbf{1} \leq \|W\|_\infty\mathbf{1}$, induction gives $(W)^\ell\mathbf{1} \leq (\|W\|_\infty)^\ell\mathbf{1}$ for every $\ell \geq 0$. Therefore, $\sum_{\ell=0}^\infty (W)^\ell\mathbf{1} \leq \sum_{\ell=0}^\infty (\|W\|_\infty)^\ell\mathbf{1} = \frac{1}{1-\|W\|_\infty}\mathbf{1}$. Note that $\widehat{\mathbf{b}} = (I_{2n} - W)^{-1}\mathbf{1} = \sum_{\ell=0}^\infty (W)^\ell\mathbf{1} \leq \frac{1}{1-\|W\|_\infty}\mathbf{1}$. So $\frac{(a-c)\beta}{4}\widehat{\mathbf{b}} \leq \frac{(a-c)\beta}{4}\frac{1}{1-\|W\|_\infty}\mathbf{1} < \frac{(a-c)\beta}{4}\frac{1}{\frac{\beta}{2}}\mathbf{1} = \frac{a-c}{2}\mathbf{1}$. Hence, $\mathbf{p}^{A*} - c\mathbf{1} = \frac{a-c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{\mathbf{b}}^B > 0$ and $\mathbf{p}^{B*} - c\mathbf{1} = \frac{a-c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{\mathbf{b}}^A > 0$. Finally, $M^{AA}, M^{BB} \succ$

0 implies these matrices have positive diagonals. Together with $M^{AA}, M^{BB} \geq 0$, this implies $\mathbf{x}^{A*} = M^{AA}\left(\frac{a-c}{2}\mathbf{1} - \frac{(a-c)\beta}{4}\widehat{\mathbf{b}}^{B}\right) > 0$. Symmetrically, we conclude that $\mathbf{x}^{B*} > 0$.

We next verify optimality against arbitrary unilateral price deviations. Let $\mathbf{p}$ and $\mathbf{x}$ stack the two products' price and consumption vectors. For any prices, consumer i's marginal utility from consumption of product s is $\frac{\partial u_i}{\partial x_i^s} = a - p_i^s - x_i^s - \beta x_i^r + \delta_s \sum_{j \in N} g_{ij}^s x_j^s$. The stacked vector of marginal utilities is $a\mathbf{1}_{2n} - \mathbf{p} - Q\mathbf{x}$. Thus, the consumer-equilibrium conditions can be written as

$$\mathbf{x} \geq \mathbf{0}, \qquad Q\mathbf{x} - a\mathbf{1}_{2n} + \mathbf{p} \geq \mathbf{0}, \qquad \mathbf{x}^\top (Q\mathbf{x} - a\mathbf{1}_{2n} + \mathbf{p}) = 0.$$

To establish existence and uniqueness, consider the auxiliary optimization problem:

$$\min_{\mathbf{x} \geq \mathbf{0}} \left\{ \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - (a\mathbf{1}_{2n} - \mathbf{p})^\top \mathbf{x} \right\}.$$

This auxiliary problem minimizes the negative of an exact potential for the consumer game at fixed prices, in the sense of Monderer and Shapley (1996). Because the objective has gradient $Q\mathbf{x} - a\mathbf{1}_{2n} + \mathbf{p}$ and Hessian Q. Its optimality conditions are therefore exactly the complementarity conditions above. As $Q \succ 0$, the objective is strictly convex and hence has at most one minimizer on the convex feasible set. Moreover, its positive definite quadratic term dominates the linear term as $\|\mathbf{x}\|_2 \to \infty$, so the objective tends to $+\infty$. A minimizer therefore exists on the nonempty closed feasible set and is unique. Consequently, every price vector induces a unique consumer equilibrium, allowing for zero consumption.

Hold firm r's price at $\mathbf{p}^{r*}$, and let $\mathbf{x}^{s\prime}$ and $\mathbf{x}^{r\prime}$ denote consumption of products s and r following any deviation by firm s. The complementarity conditions for the rival's product imply $Q^r \mathbf{x}^{r\prime} \geq a\mathbf{1} - \mathbf{p}^{r*} - \beta \mathbf{x}^{s\prime}$. As $K^r \geq 0$, it follows that $\mathbf{x}^{r\prime} \geq K^r(a\mathbf{1} - \mathbf{p}^{r*} - \beta \mathbf{x}^{s\prime})$. Product s's complementarity condition $(\mathbf{x}^{s\prime})^\top \left(Q^s \mathbf{x}^{s\prime} + \beta \mathbf{x}^{r\prime} - a\mathbf{1} + \mathbf{p}^s\right) = 0$ therefore gives

$$\begin{aligned}
\pi^s(\mathbf{p}^s, \mathbf{p}^{r*}) &= (a-c)\mathbf{1}^\top \mathbf{x}^{s\prime} - (\mathbf{x}^{s\prime})^\top Q^s \mathbf{x}^{s\prime} - \beta (\mathbf{x}^{s\prime})^\top \mathbf{x}^{r\prime} \\
&\leq (a-c)\mathbf{1}^\top \mathbf{x}^{s\prime} - (\mathbf{x}^{s\prime})^\top Q^s \mathbf{x}^{s\prime} - \beta (\mathbf{x}^{s\prime})^\top K^r (a\mathbf{1} - \mathbf{p}^{r*} - \beta \mathbf{x}^{s\prime}) \\
&= \left[(a-c)\mathbf{1} - \beta K^r (a\mathbf{1} - \mathbf{p}^{r*})\right]^\top \mathbf{x}^{s\prime} - (\mathbf{x}^{s\prime})^\top (Q^s - \beta^2 K^r) \mathbf{x}^{s\prime} \\
&= \pi^{s*} - (\mathbf{x}^{s\prime} - \mathbf{x}^{s*})^\top (Q^s - \beta^2 K^r)(\mathbf{x}^{s\prime} - \mathbf{x}^{s*}).
\end{aligned}$$

The third equality uses the candidate's consumer equations and pricing first-order condition, which together imply $(a-c)\mathbf{1}-\beta K^r(a\mathbf{1}-\mathbf{p}^{r*})=2(Q^s-\beta^2K^r)\mathbf{x}^{s*}$. Then we conclude $\pi^s(\mathbf{p}^s,\mathbf{p}^{r*})\le\pi^{s*}$ due to $Q^s-\beta^2K^r=(M^{ss})^{-1}\succ 0$. Hence, the candidate is a Nash equilibrium even when deviations inducing zero consumption are allowed.

To establish uniqueness, we show that every Nash equilibrium must induce strictly positive consumption. Let $q:=1-\max\{\delta_A d^A_{\max},\delta_B d^B_{\max}\}$, so Assumption 2 implies $0\le\beta<q$. For each $s\in\{A,B\}$, $\delta_s G^s\mathbf{1}\le(1-q)\mathbf{1}$. The Neumann-series representation $K^s=\sum_{\ell=0}^{\infty}(\delta_s G^s)^\ell$ therefore gives $K^s\mathbf{1}\le\frac{1}{q}\mathbf{1}$.

Fix the rival's price $\mathbf{p}^r$ and, for any $\mathbf{x}^{s\prime}\ge\mathbf{0}$, let $\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})$ denote consumption of the rival's product, determined by

$$\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})=\max\left\{\mathbf{0},\,a\mathbf{1}-\mathbf{p}^r-\beta\mathbf{x}^{s\prime}+\delta_r G^r\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})\right\},\tag{A.4}$$

where the maximum is taken componentwise. Because prices can take any real values, firm s can implement any such $\mathbf{x}^{s\prime}$ by setting $\mathbf{p}^s=a\mathbf{1}-Q^s\mathbf{x}^{s\prime}-\beta\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})$. Its resulting profit is $\widehat{\pi}^s(\mathbf{x}^{s\prime};\mathbf{p}^r)=(a-c)\mathbf{1}^\top\mathbf{x}^{s\prime}-(\mathbf{x}^{s\prime})^\top Q^s\mathbf{x}^{s\prime}-\beta(\mathbf{x}^{s\prime})^\top\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})$. Hence, at any Nash equilibrium, its product's consumption vector must maximize $\widehat{\pi}^s(\mathbf{x}^{s\prime};\mathbf{p}^r)$ over $\mathbf{x}^{s\prime}\ge\mathbf{0}$.

As $\|\delta_r G^r\|_\infty<1$, the fixed-point equation (A.4) is a contraction in the infinity norm. The Banach fixed-point theorem therefore ensures that iteration from $\mathbf{0}$ converges to the unique fixed point. Comparing the iterates gives $\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime}+\varepsilon\mathbf{e}_i)\le\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})$ for $\varepsilon>0$, where $\mathbf{e}_i$ is the ith unit vector. For $0<\varepsilon\le x_i^{s\prime}$, it also gives

$$\mathbf{0}\le\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime}-\varepsilon\mathbf{e}_i)-\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})\le\beta\varepsilon K^r\mathbf{e}_i.$$

Indeed, monotonicity gives the first inequality, and (A.4) implies $\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime}-\varepsilon\mathbf{e}_i)-\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})\le\beta\varepsilon\mathbf{e}_i+\delta_r G^r[\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime}-\varepsilon\mathbf{e}_i)-\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime})]$. Rearranging and premultiplying by $K^r\ge 0$ yields the upper bound.

Now consider any Nash equilibrium, denote its consumption vectors by $\mathbf{x}^{A\prime}$ and $\mathbf{x}^{B\prime}$, and let $X:=\max_{s,i}x_i^{s\prime}$. If $X>0$, choose a product s and consumer i with $x_i^{s\prime}=X$.

As $x_j^{s\prime}\le X$ for all j and $\delta_s\sum_j g_{ij}^s\le 1-q$, we have $(Q^s\mathbf{x}^{s\prime})_i=X-\delta_s\sum_j g_{ij}^s x_j^{s\prime}\ge X-(1-q)X=qX$. Moreover, symmetry of K^r and $K^r\mathbf{1}\le\frac{1}{q}\mathbf{1}$ give $(\mathbf{x}^{s\prime})^\top K^r\mathbf{e}_i=\sum_j x_j^{s\prime}K^r_{ji}\le\frac{X}{q}$. For $0<\varepsilon<X$, optimality and

the preceding response bound therefore imply

$$\begin{aligned}0 \geq \widehat{\pi}^s(\mathbf{x}^{s\prime} - \varepsilon \mathbf{e}_i; \mathbf{p}^r) - \widehat{\pi}^s(\mathbf{x}^{s\prime}; \mathbf{p}^r) &= -\varepsilon(a-c) + 2\varepsilon(Q^s \mathbf{x}^{s\prime})_i - \varepsilon^2 - \beta(\mathbf{x}^{s\prime})^\top \left[\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime} - \varepsilon \mathbf{e}_i) - \mathbf{x}^{r\prime}\right] + \beta \varepsilon x_i^{r\prime}(\mathbf{x}^{s\prime} - \varepsilon \mathbf{e}_i) \\ &\geq \varepsilon\left[-(a-c) + 2(Q^s \mathbf{x}^{s\prime})_i + \beta x_i^{r\prime}(x^{s\prime}) - \beta^2 (\mathbf{x}^{s\prime})^\top K^r \mathbf{e}_i\right] - \varepsilon^2 \\ &\geq \varepsilon\left[\left(2q - \frac{\beta^2}{q}\right) X - (a-c)\right] - \varepsilon^2,\end{aligned}$$

where the second inequality uses $\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime} - \varepsilon \mathbf{e}_i) - \mathbf{x}^{r\prime} \leq \beta \varepsilon K^r \mathbf{e}_i$ and $\beta \varepsilon x_i^{r\prime}(\mathbf{x}^{s\prime} - \varepsilon \mathbf{e}_i) \geq \beta \varepsilon x_i^{r\prime}(x^{s\prime})$, the third inequality uses $(Q^s \mathbf{x}^{s\prime})_i \geq qX, \beta x_i^{r\prime} \geq 0, (\mathbf{x}^{s\prime})^\top K^r \mathbf{e}_i \leq \frac{X}{q}$. Letting ε tend to zero yields $X \leq \frac{a-c}{2q - \frac{\beta^2}{q}} < \frac{a-c}{q}$. The same bound holds trivially when $X = 0$. In either case, $\beta X < a - c$.

If some consumption choice satisfies $x_i^{s\prime} = 0$, then $(Q^s \mathbf{x}^{s\prime})_i \leq 0$. As increasing $x_i^{s\prime}$ weakly reduces the rival product's consumption, for sufficiently small $\varepsilon > 0$ we obtain

$$\begin{aligned}\widehat{\pi}^s(\mathbf{x}^{s\prime} + \varepsilon \mathbf{e}_i; \mathbf{p}^r) - \widehat{\pi}^s(\mathbf{x}^{s\prime}; \mathbf{p}^r) &= \varepsilon(a-c) - 2\varepsilon(Q^s \mathbf{x}^{s\prime})_i - \varepsilon^2 - \beta(\mathbf{x}^{s\prime})^\top \left[\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime} + \varepsilon \mathbf{e}_i) - \mathbf{x}^{r\prime}(x^{s\prime})\right] - \beta \varepsilon x_i^{r\prime}(\mathbf{x}^{s\prime} + \varepsilon \mathbf{e}_i) \\ &\geq \varepsilon\left[(a-c) - 2(Q^s \mathbf{x}^{s\prime})_i - \beta x_i^{r\prime}(x^{s\prime})\right] - \varepsilon^2 \\ &\geq \varepsilon\left[(a-c) - \beta X\right] - \varepsilon^2 > 0,\end{aligned}$$

where the first inequality uses $\mathbf{x}^{r\prime}(\mathbf{x}^{s\prime} + \varepsilon \mathbf{e}_i) \leq \mathbf{x}^{r\prime}(x^{s\prime})$ and the last inequality uses $\beta X < a - c$, thus contradicting optimality.

Thus, every Nash equilibrium induces strictly positive consumption and must satisfy the pricing first-order conditions derived above. Since those conditions have a unique solution, the candidate is the unique Nash equilibrium of the unrestricted price game. □

Proof of Corollary 1. If $\beta = 0$, Theorem 1 immediately gives $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1}$ for $s \in \{A, B\}$.

Moreover, both Q and $I_{2n} - W$ are block diagonal, with diagonal blocks $I - \delta_s G^s$. Hence, the definitions of $M = Q^{-1}$ and $\widehat{\mathbf{b}} = (I_{2n} - W)^{-1}\mathbf{1}_{2n}$ imply $M^{ss} = (I - \delta_s G^s)^{-1}, \widehat{\mathbf{b}}^s = (I - \delta_s G^s)^{-1}\mathbf{1}$. Substituting $\beta = 0$ into Theorem 1 therefore yields

$$\mathbf{x}^{s*} = \frac{a-c}{2} M^{ss}\mathbf{1} = \frac{a-c}{2}(I - \delta_s G^s)^{-1}\mathbf{1} = \frac{a-c}{2}\widehat{\mathbf{b}}^s, \qquad s \in \{A, B\}.$$

Now suppose that $G^A = G^B = G$ and $\delta_A = \delta_B = \delta$. Therefore, $I_{2n} - W = \begin{pmatrix} I - \delta G & -\frac{\beta}{2} I \\ -\frac{\beta}{2} I & I - \delta G \end{pmatrix}$. By the

definition of centrality, $\begin{pmatrix} I-\delta G & -\frac{\beta}{2}I \\ -\frac{\beta}{2}I & I-\delta G \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{b}}^A \\ \widehat{\mathbf{b}}^B \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$. Thus we get

$$\begin{cases} (I-\delta G)\widehat{\mathbf{b}}^A - \dfrac{\beta}{2}\widehat{\mathbf{b}}^B = \mathbf{1}, \\ (I-\delta G)\widehat{\mathbf{b}}^B - \dfrac{\beta}{2}\widehat{\mathbf{b}}^A = \mathbf{1}. \end{cases}$$

By symmetry, $\widehat{\mathbf{b}}^A = \widehat{\mathbf{b}}^B$, so the equations reduce to $\left[\left(1-\frac{\beta}{2}\right)I - \delta G\right]\widehat{\mathbf{b}}^A = \mathbf{1}$. Hence, $\widehat{\mathbf{b}}^A = \widehat{\mathbf{b}}^B = 2\left[(2-\beta)I - 2\delta G\right]^{-1}\mathbf{1}$. Substituting the common centrality vector into Theorem 1 gives

$$\mathbf{p}^{A*} = \mathbf{p}^{B*} = \mathbf{p}^* = \frac{a+c}{2}\mathbf{1} - \frac{(a-c)\beta}{2}\left[(2-\beta)I - 2\delta G\right]^{-1}\mathbf{1}.$$

□

Proof of Theorem 2. For the network pair G^A and $G^B(P) = PG^BP^\top$, write

$$K^A = (I-\delta_A G^A)^{-1}, \qquad K_P^B = (I-\delta_B PG^BP^\top)^{-1} = PK^BP^\top.$$

Both matrices are symmetric and satisfy $K^A\mathbf{1} = \mathbf{b}^A$ and $K_P^B\mathbf{1} = P\mathbf{b}^B$. As permutations preserve the maximum weighted degree, every configuration considered below satisfies Assumption 2 whenever $0 \le \beta < 1 - \max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\}$.

All remainder terms below are evaluated as $\beta \downarrow 0$, holding the remaining primitives and the permutation fixed. We first compare equilibrium average prices. Using the representation of multiplex centrality derived in footnote 17, together with $K^A\mathbf{1} = \mathbf{b}^A$ and $K_P^B\mathbf{1} = P\mathbf{b}^B$, we obtain

$$\begin{cases} \widehat{\mathbf{b}}^A(P,\beta) = \mathbf{b}^A + \dfrac{\beta}{2}K^A\widehat{\mathbf{b}}^B(P,\beta), \\ \widehat{\mathbf{b}}^B(P,\beta) = P\mathbf{b}^B + \dfrac{\beta}{2}K_P^B\widehat{\mathbf{b}}^A(P,\beta). \end{cases}$$

Solving the system yields

$$\begin{cases} \widehat{\mathbf{b}}^A(P,\beta) = \left(I - \dfrac{\beta^2}{4} K^A K_P^B\right)^{-1} \left(\mathbf{b}^A + \dfrac{\beta}{2} K^A P \mathbf{b}^B\right), \\ \widehat{\mathbf{b}}^B(P,\beta) = \left(I - \dfrac{\beta^2}{4} K_P^B K^A\right)^{-1} \left(P\mathbf{b}^B + \dfrac{\beta}{2} K_P^B \mathbf{b}^A\right). \end{cases}$$

For sufficiently small β, the Neumann series gives $\left(I - \frac{\beta^2}{4} K^A K_P^B\right)^{-1} = I + O(\beta^2)$, and the same expansion holds with $K^A K_P^B$ replaced by $K_P^B K^A$. Consequently,

$$\begin{cases} \widehat{\mathbf{b}}^A(P,\beta) = \mathbf{b}^A + \dfrac{\beta}{2} K^A P \mathbf{b}^B + O(\beta^2), \\ \widehat{\mathbf{b}}^B(P,\beta) = P\mathbf{b}^B + \dfrac{\beta}{2} K_P^B \mathbf{b}^A + O(\beta^2). \end{cases}$$

Substituting into Theorem 1 yields

$$\begin{cases} \mathbf{p}^{A*}(P,\beta) = \dfrac{a+c}{2}\mathbf{1} - \dfrac{(a-c)\beta}{4} P\mathbf{b}^B - \dfrac{(a-c)\beta^2}{8} K_P^B \mathbf{b}^A + O(\beta^3), \\ \mathbf{p}^{B*}(P,\beta) = \dfrac{a+c}{2}\mathbf{1} - \dfrac{(a-c)\beta}{4} \mathbf{b}^A - \dfrac{(a-c)\beta^2}{8} K^A P \mathbf{b}^B + O(\beta^3). \end{cases}$$

We next average these prices across consumers. Because $P^\top \mathbf{1} = \mathbf{1}$, we have $\mathbf{1}^\top P\mathbf{b}^B = \mathbf{1}^\top \mathbf{b}^B$. Moreover, symmetry of K^A and K_P^B gives

$$\begin{cases} \mathbf{1}^\top K_P^B \mathbf{b}^A = (P\mathbf{b}^B)^\top \mathbf{b}^A = S(P), \\ \mathbf{1}^\top K^A P\mathbf{b}^B = (\mathbf{b}^A)^\top P\mathbf{b}^B = S(P). \end{cases}$$

Thus, for $s, r \in \{A, B\}$ with $r \neq s$,

$$\overline{p}^{s*}(P,\beta) = \frac{a+c}{2} - \frac{(a-c)\beta}{4n} \mathbf{1}^\top \mathbf{b}^r - \frac{(a-c)\beta^2}{8n} S(P) + O(\beta^3). \tag{A.5}$$

The constant and first-order terms are independent of P. The first term distinguishing average prices across assignments is therefore the second-order term involving $S(P)$.

We now compare equilibrium profits. Theorem 1 implies $\pi^{s*} = (\mathbf{p}^{s*} - c\mathbf{1})^\top M^{ss} (\mathbf{p}^{s*} - c\mathbf{1})$ for $s \in \{A, B\}$, where the arguments (P, β) are suppressed for brevity.

By the Neumann series,

$$\begin{cases} M^{AA} = \left[(K^A)^{-1} - \beta^2 K_P^B\right]^{-1} = (I - \beta^2 K^A K_P^B)^{-1} K^A = K^A + O(\beta^2), \\ M^{BB} = \left[(K_P^B)^{-1} - \beta^2 K^A\right]^{-1} = (I - \beta^2 K_P^B K^A)^{-1} K_P^B = K_P^B + O(\beta^2). \end{cases}$$

The price expansions above give

$$\begin{cases} \mathbf{p}^{A*} - c\mathbf{1} = \dfrac{a-c}{2}\mathbf{1} - \dfrac{(a-c)\beta}{4} P\mathbf{b}^B + O(\beta^2), \\ \mathbf{p}^{B*} - c\mathbf{1} = \dfrac{a-c}{2}\mathbf{1} - \dfrac{(a-c)\beta}{4} \mathbf{b}^A + O(\beta^2). \end{cases}$$

Substituting into firm A's profit expression gives

$$\begin{aligned} \pi^{A*}(P,\beta) &= \frac{(a-c)^2}{4}\mathbf{1}^\top K^A \mathbf{1} - \frac{(a-c)^2\beta}{8}\left[\mathbf{1}^\top K^A P\mathbf{b}^B + (P\mathbf{b}^B)^\top K^A \mathbf{1}\right] + O(\beta^2) \\ &= \frac{(a-c)^2}{4}\mathbf{1}^\top \mathbf{b}^A - \frac{(a-c)^2\beta}{4} S(P) + O(\beta^2). \end{aligned}$$

The last equality uses $K^A\mathbf{1} = \mathbf{b}^A$ and $\mathbf{1}^\top K^A P\mathbf{b}^B = (P\mathbf{b}^B)^\top K^A\mathbf{1} = S(P)$. For firm B, the same calculation applies because $\mathbf{1}^\top K_P^B\mathbf{1} = \mathbf{1}^\top P\mathbf{b}^B = \mathbf{1}^\top\mathbf{b}^B$ and $\mathbf{1}^\top K_P^B\mathbf{b}^A = (\mathbf{b}^A)^\top K_P^B\mathbf{1} = S(P)$. Thus, for each $s \in \{A,B\}$,

$$\pi^{s*}(P,\beta) = \frac{(a-c)^2}{4}\mathbf{1}^\top\mathbf{b}^s - \frac{(a-c)^2\beta}{4} S(P) + O(\beta^2). \tag{A.6}$$

Although the baseline terms can differ across firms, they are invariant to P.

Finally, let $\Delta S := S(P_1) - S(P_0) > 0$. Taking differences across the two assignments in (A.5) and (A.6) yields

$$\begin{cases} \overline{p}^{s*}(P_1,\beta) - \overline{p}^{s*}(P_0,\beta) = -\dfrac{(a-c)\beta^2}{8n}\Delta S + O(\beta^3), \\ \pi^{s*}(P_1,\beta) - \pi^{s*}(P_0,\beta) = -\dfrac{(a-c)^2\beta}{4}\Delta S + O(\beta^2). \end{cases}$$

After division by β^2 and β, respectively, the normalized differences converge to $-\frac{a-c}{8n}\Delta S < 0$ and $-\frac{(a-c)^2}{4}\Delta S < 0$ as $\beta \downarrow 0$. Hence, for each $s \in \{A,B\}$, there exist $\varepsilon_s, \eta_s > 0$ such that

$$\begin{cases} \overline{p}^{s*}(P_1,\beta) - \overline{p}^{s*}(P_0,\beta) < -\dfrac{(a-c)\beta^2}{16n}\Delta S < 0, \quad 0 < \beta < \varepsilon_s, \\ \pi^{s*}(P_1,\beta) - \pi^{s*}(P_0,\beta) < -\dfrac{(a-c)^2\beta}{8}\Delta S < 0, \quad 0 < \beta < \eta_s. \end{cases}$$

Choosing $\overline{\beta} = \min\left\{\varepsilon_A, \varepsilon_B, \eta_A, \eta_B, \frac{1-\max\{\delta_A d^A_{\max}, \delta_B d^B_{\max}\}}{2}\right\} > 0$ establishes the results. □

Proof of Proposition 2. As we maintain Assumption 2, the results in Theorem 1 and Proposition 1 still holds.

Proof of part (i). Taking derivatives of the price formula in Theorem 1 with respect to a gives $\frac{\partial \mathbf{p}^{s*}}{\partial a} = \frac{1}{2}\mathbf{1} - \frac{\beta}{4}\widehat{b}^r$, for $s, r \in \{A, B\}$ with $r \neq s$. The price formula in Theorem 1 also establishes that $-\frac{\beta}{4}\widehat{b}^r = \frac{p^{s*}-c\mathbf{1}}{a-c} - \frac{1}{2}\mathbf{1}$. Combining these identities yields $\frac{\partial \mathbf{p}^{s*}}{\partial a} = \frac{\mathbf{p}^{s*}-c\mathbf{1}}{a-c} > 0$ where the inequality follows from the interiority implied by Theorem 1. Differentiating the price formula with respect to c gives $\frac{\partial \mathbf{p}^{s*}}{\partial c} = \frac{1}{2}\mathbf{1} + \frac{\beta}{4}\widehat{b}^r > 0$ where the inequality is obvious by noticing that $\forall i \in N, \widehat{b}^r_i \geq 0$.

By Theorem 1, the equilibrium consumption vector satisfies $\mathbf{x}^{s*} = (a-c)M^{ss}\left(\frac{1}{2}\mathbf{1} - \frac{\beta}{4}\widehat{\mathbf{b}}^r\right)$. Differentiation gives $\frac{\partial \mathbf{x}^{s*}}{\partial a} = -\frac{\partial \mathbf{x}^{s*}}{\partial c} = \frac{\mathbf{x}^{s*}}{a-c} > 0$, where the inequality follows from $\mathbf{x}^{s*} > 0$ implied by Theorem 1.

Proof of part (ii). Differentiating $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}$ with respect to β yields $\frac{\partial \widehat{\mathbf{b}}}{\partial \beta} = (I_{2n} - W)^{-1}\frac{\partial W}{\partial \beta}\widehat{\mathbf{b}} \geq 0$. Therefore, $\frac{\partial \mathbf{p}^{s*}}{\partial \beta} = -\frac{a-c}{4}\widehat{\mathbf{b}}^r - \frac{(a-c)\beta}{4}\frac{\partial \widehat{\mathbf{b}}^r}{\partial \beta} < 0$.

However, greater substitutability can either increase or decrease equilibrium consumption. To illustrate, consider $G^A = G^B = 0$. Theorem 1 and consumer equilibrium give, $\forall i, s$, $x_i^{s*} = \frac{a-c}{(1+\beta)(2-\beta)}$, $\frac{\mathrm{d}x_i^{s*}}{\mathrm{d}\beta} = \frac{(a-c)(2\beta-1)}{(1+\beta)^2(2-\beta)^2}$. The derivative is negative for $0 \leq \beta < \frac{1}{2}$ and positive for $\frac{1}{2} < \beta < 1$.

Proof of part (iii). Consider a change in δ_A; the analysis for δ_B is symmetric.

The multiplex centrality satisfies $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}$. Differentiating both sides with respect to δ_A gives $-\frac{\partial W}{\partial \delta_A}\widehat{\mathbf{b}} + (I_{2n} - W)\frac{\partial \widehat{\mathbf{b}}}{\partial \delta_A} = \mathbf{0}$. Rearranging the equation yields $\frac{\partial \widehat{\mathbf{b}}}{\partial \delta_A} = (I_{2n} - W)^{-1}\begin{pmatrix} G^A\widehat{\mathbf{b}}^A \\ \mathbf{0} \end{pmatrix} \geq \mathbf{0}$. The inequality follows because, under Assumption 2, $(I_{2n} - W)^{-1} \geq 0$, $G^A \geq 0$, and $\widehat{\mathbf{b}}^A \geq 0$.

Theorem 1 consequently implies $\frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} = -\frac{(a-c)\beta}{4}\frac{\partial \widehat{\mathbf{b}}^B}{\partial \delta_A} \leq 0$, $\frac{\partial \mathbf{p}^{B*}}{\partial \delta_A} = -\frac{(a-c)\beta}{4}\frac{\partial \widehat{\mathbf{b}}^A}{\partial \delta_A} \leq 0$.

To analyze equilibrium consumption, we first compute the derivative of $M = Q^{-1}$ with respect to δ_A. Differentiation gives $\frac{\partial M}{\partial \delta_A} = -M\frac{\partial Q}{\partial \delta_A}M$. Using $\frac{\partial Q}{\partial \delta_A} = -\operatorname{diag}(G^A, 0)$, we obtain

$$\frac{\partial M}{\partial \delta_A} = M\operatorname{diag}(G^A, 0)M = \begin{pmatrix} M^{AA}G^AM^{AA} & M^{AA}G^AM^{AB} \\ M^{BA}G^AM^{AA} & M^{BA}G^AM^{AB} \end{pmatrix}. \tag{A.7}$$

Writing $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}$ in block form gives

$$\begin{cases} Q^A \widehat{\mathbf{b}}^A - \dfrac{\beta}{2}\widehat{\mathbf{b}}^B = \mathbf{1}, \\ Q^B \widehat{\mathbf{b}}^B - \dfrac{\beta}{2}\widehat{\mathbf{b}}^A = \mathbf{1}. \end{cases}$$

Combining these equations with Theorem 1 yields $a\mathbf{1} - \mathbf{p}^{s*} = \frac{a-c}{2}\left(\mathbf{1} + \frac{\beta}{2}\widehat{\mathbf{b}}^r\right) = \frac{a-c}{2}Q^s\widehat{\mathbf{b}}^s$ for $s, r \in \{A, B\}$ with $s \neq r$.

As $\operatorname{diag}(Q^A, Q^B) = \frac{1}{3}Q + \frac{2}{3}(I_{2n} - W)$, Proposition 1 implies

$$\begin{aligned} \begin{pmatrix} \mathbf{x}^{A*} \\ \mathbf{x}^{B*} \end{pmatrix} &= M \begin{pmatrix} a\mathbf{1} - \mathbf{p}^{A*} \\ a\mathbf{1} - \mathbf{p}^{B*} \end{pmatrix} = \frac{a-c}{2} M \operatorname{diag}(Q^A, Q^B)\widehat{\mathbf{b}} \\ &= \frac{a-c}{2} M \left[\frac{1}{3}Q + \frac{2}{3}(I_{2n} - W)\right]\widehat{\mathbf{b}} = \frac{a-c}{6}\widehat{\mathbf{b}} + \frac{a-c}{3}M\mathbf{1}, \end{aligned} \tag{A.8}$$

where the last equality uses $MQ = I_{2n}$ and $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}$.

Differentiating this expression with respect to δ_A yields

$$\begin{aligned} \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^A}{\partial \delta_A} + \frac{a-c}{3}\left(\frac{\partial M^{AA}}{\partial \delta_A} + \frac{\partial M^{AB}}{\partial \delta_A}\right)\mathbf{1} \\ &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^A}{\partial \delta_A} + \frac{a-c}{3}\left(M^{AA}G^A M^{AA} + M^{AA}G^A M^{AB}\right)\mathbf{1} \\ &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^A}{\partial \delta_A} + \frac{a-c}{3}M^{AA}G^A(M^{AA} + M^{AB})\mathbf{1}. \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A} &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^B}{\partial \delta_A} + \frac{a-c}{3}\left(\frac{\partial M^{BA}}{\partial \delta_A} + \frac{\partial M^{BB}}{\partial \delta_A}\right)\mathbf{1} \\ &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^B}{\partial \delta_A} + \frac{a-c}{3}\left(M^{BA}G^A M^{AA} + M^{BA}G^A M^{AB}\right)\mathbf{1} \\ &= \frac{a-c}{6}\frac{\partial \widehat{\mathbf{b}}^B}{\partial \delta_A} + \frac{a-c}{3}M^{BA}G^A(M^{AA} + M^{AB})\mathbf{1}. \end{aligned}$$

We now establish

$$(M^{ss} + M^{sr})\mathbf{1} > 0, \qquad s, r \in \{A, B\},\ s \neq r. \tag{A.9}$$

As $G^r\mathbf{1} \leq d^r_{\max}\mathbf{1}$ and $G^r \geq 0$, induction gives $(G^r)^\ell\mathbf{1} \leq (d^r_{\max})^\ell\mathbf{1}$ for every $\ell \geq 0$. The Neumann-series

representation therefore implies

$$0 \leq K^r \mathbf{1} = \sum_{\ell=0}^{\infty} \delta_r^{\ell} (G^r)^{\ell} \mathbf{1} \leq \sum_{\ell=0}^{\infty} (\delta_r d_{\max}^r)^{\ell} \mathbf{1} = \frac{1}{1-\delta_r d_{\max}^r} \mathbf{1} \tag{A.10}$$

Using $M^{sr} = -\beta M^{ss} K^r$, we obtain

$$(M^{ss} + M^{sr})\mathbf{1} = M^{ss}(\mathbf{1} - \beta K^r \mathbf{1}) \geq \left(1 - \frac{\beta}{1-\delta_r d_{\max}^r}\right) M^{ss} \mathbf{1} > 0,$$

where the last inequality follows from Assumption 2 and the fact that M^{ss} is componentwise nonnegative with a positive diagonal.

Equipped with this inequality, we conclude that the derivative of own-product consumption is nonnegative because $a - c > 0$, $\frac{\partial \widehat{\mathbf{b}}^A}{\partial \delta_A} \geq 0$, $M^{AA} \geq 0$, $G^A \geq 0$ and $(M^{AA} + M^{AB})\mathbf{1} > 0$. Moreover, if $G^A \neq 0$, the second term in that equation has a strictly positive component, because $(M^{AA} + M^{AB})\mathbf{1} > 0$ and M^{AA} has positive diagonal. Thus aggregate consumption of product A strictly increases.

However, regarding rival-product consumption, the first term is nonnegative, whereas the second is nonpositive because $M^{BA} \leq 0$. So in general, $\frac{\partial \mathbf{x}^{B*}}{\partial \delta_A}$ does not have a determinate sign. For a direct example, suppose both networks connect consumers 1 and 2 with a unit-weight link, and let $\beta = \frac{1}{2}$ and $\delta_B = \frac{9}{20}$. The equilibrium conditions give $\frac{\partial x_i^{B*}}{\partial \delta_A} = \frac{10(a-c)(9-44\delta_A)(29-44\delta_A)}{(39-44\delta_A)^2(6-11\delta_A)^2}, i = 1,2$. This derivative is positive for $0 < \delta_A < \frac{9}{44}$ and negative for $\frac{9}{44} < \delta_A < \frac{1}{2}$.

Nevertheless we can determine this sign in a special case. At $\beta = 0$, Q is block diagonal, so $M^{BA} = 0$ and $M^{BB} = K^B$. The centrality equation also gives $\widehat{\mathbf{b}}^B = K^B \mathbf{1}$. Thus, (A.8) reduces to $\mathbf{x}^{B*} = \frac{a-c}{2} K^B \mathbf{1}$. Because $K^B = (I_n - \delta_B G^B)^{-1}$ is independent of δ_A, holding all other primitives fixed yields $\left.\frac{\partial \mathbf{x}^{B*}}{\partial \delta_A}\right|_{\beta=0} = \mathbf{0}$. So equilibrium consumption of the rival's product is unaffected by a change in δ_A at $\beta = 0$.

Proof of part (iv). As the argument is symmetric across networks, it suffices to consider an expansion of firm A's network. For $G^{A\prime} \geq G^A$, consider $G^A(t) = G^A + t(G^{A\prime} - G^A)$, $t \in [0,1]$, holding G^B and all other primitives fixed.

For every $t \in [0,1]$, the matrix $G^A(t)$ is symmetric, componentwise nonnegative, and has a zero diagonal. Moreover, $\sum_j G_{ij}^A(t) = (1-t)\sum_j G_{ij}^A + t\sum_j G_{ij}^{A\prime} \leq \sum_j G_{ij}^{A\prime}$ for every i. Hence,

$$\beta + \max\left\{\delta_A \|G^A(t)\|_\infty, \delta_B \|G^B\|_\infty\right\} \leq \beta + \max\left\{\delta_A \|G^{A\prime}\|_\infty, \delta_B \|G^B\|_\infty\right\} < 1,$$

which means Assumption 2 holds along the path. All equilibrium objects below are evaluated at $(G^A(t), G^B)$; we suppress their dependence on t where convenient.

By the definitions of W and Q, $\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}Q}{\mathrm{d}t} = \operatorname{diag}\big(\delta_A(G^{A\prime} - G^A), 0\big)$. Differentiating $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}_{2n}$ gives

$$\frac{\mathrm{d}\widehat{\mathbf{b}}}{\mathrm{d}t} = (I_{2n} - W)^{-1} \begin{pmatrix} \delta_A(G^{A\prime} - G^A)\widehat{\mathbf{b}}^A \\ \mathbf{0} \end{pmatrix} \geq \mathbf{0}.$$

Theorem 1 therefore implies $\frac{\mathrm{d}\mathbf{p}^{s*}}{\mathrm{d}t} = -\frac{(a-c)\beta}{4}\frac{\mathrm{d}\widehat{\mathbf{b}}^r}{\mathrm{d}t} \leq \mathbf{0}$ for $s, r \in \{A, B\}$ with $s \neq r$. Integrating over $t \in [0,1]$ shows that expanding firm A's network weakly lowers equilibrium prices for both products.

To analyze equilibrium consumption, differentiating $QM = I_{2n}$ yields $\frac{\mathrm{d}M}{\mathrm{d}t} = -M\frac{\mathrm{d}Q}{\mathrm{d}t}M = \delta_A M \operatorname{diag}(G^{A\prime} - G^A, 0)M$. Differentiating the A-block of (A.8) therefore yields

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}t} &= \frac{a-c}{6}\frac{\mathrm{d}\widehat{\mathbf{b}}^A}{\mathrm{d}t} + \frac{a-c}{3}\left(\frac{\mathrm{d}M^{AA}}{\mathrm{d}t} + \frac{\mathrm{d}M^{AB}}{\mathrm{d}t}\right)\mathbf{1} \\ &= \frac{a-c}{6}\frac{\mathrm{d}\widehat{\mathbf{b}}^A}{\mathrm{d}t} + \frac{(a-c)\delta_A}{3}M^{AA}(G^{A\prime} - G^A)(M^{AA} + M^{AB})\mathbf{1} \geq 0. \end{aligned}$$

The inequality follows from $\frac{\mathrm{d}\widehat{\mathbf{b}}^A}{\mathrm{d}t} \geq \mathbf{0}$, $M^{AA} \geq 0$, $G^{A\prime} - G^A \geq 0$, and $(M^{AA} + M^{AB})\mathbf{1} > \mathbf{0}$ by (A.9). Integrating over $t \in [0,1]$ shows that expanding firm A's network weakly increases each consumer's equilibrium consumption of its product.

If $\delta_A > 0$ and $G^{A\prime} \neq G^A$, the second term above is nonnegative and nonzero, as $(M^{AA} + M^{AB})\mathbf{1} > \mathbf{0}$ and $M^{AA} \geq 0$ has a strictly positive diagonal. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{1}^\top \mathbf{x}^{A*}) > 0$ along the path, implying a strict increase in aggregate consumption of product A.

Consumption of the rival's product can move in either direction even when adding a single link to network A. Consider two consumers and fix $\beta = \frac{1}{2}$, $\delta_A = \frac{3}{8}$, and $\delta_B = \frac{9}{20}$. Initially, network A has no links, network B connects the two consumers with a unit-weight link. For each $i = 1, 2$, adding a link of weight $w = \frac{2}{3}$ gives $\Delta x_i^{B*} = \frac{5(a-c)}{819} > 0$, whereas adding a link of weight $w = 1$ gives $\Delta x_i^{B*} = -\frac{7(a-c)}{351} < 0$. □

Proof of Proposition 3. Proof of part (i). We first examine the effects of valuation a and marginal cost c on equilibrium profits. Theorem 1 implies $\pi^{s*} = \frac{(a-c)^2}{4}\left(\mathbf{1} - \frac{\beta}{2}\widehat{\mathbf{b}}^r\right)^\top M^{ss}\left(\mathbf{1} - \frac{\beta}{2}\widehat{\mathbf{b}}^r\right)$. Hence $\frac{\partial \pi^{s*}}{\partial a} = -\frac{\partial \pi^{s*}}{\partial c} = \frac{2\pi^{s*}}{a-c} > 0$, because $\mathbf{p}^{s*} > c\mathbf{1}$ together with $x^{s*} > 0$ implies $\pi^{s*} > 0$.

We next turn to substitutability. Differentiate $\pi^{s*} = (\mathbf{p}^{s*} - c\mathbf{1})^\top \mathbf{x}^{s*}$ with respect to β, holding all other

primitives fixed. By the product rule and Proposition 1,

$$\begin{aligned}\frac{\partial \pi^{s*}}{\partial \beta} &= \left(\frac{\partial \mathbf{p}^{s*}}{\partial \beta}\right)^{\top} \mathbf{x}^{s*} + (\mathbf{p}^{s*} - c\mathbf{1})^{\top} \left[\frac{\partial M^{ss}}{\partial \beta}(a\mathbf{1} - \mathbf{p}^{s*}) + \frac{\partial M^{sr}}{\partial \beta}(a\mathbf{1} - \mathbf{p}^{r*})\right] \\ &\quad - (\mathbf{p}^{s*} - c\mathbf{1})^{\top} \left[M^{ss}\frac{\partial \mathbf{p}^{s*}}{\partial \beta} + M^{sr}\frac{\partial \mathbf{p}^{r*}}{\partial \beta}\right] \\ &= \left(\frac{\partial \mathbf{p}^{s*}}{\partial \beta}\right)^{\top} \mathbf{x}^{s*} + (\mathbf{p}^{s*} - c\mathbf{1})^{\top} \left.\frac{\partial \mathbf{x}^{s}}{\partial \beta}\right|_{\mathbf{p}^{s},\mathbf{p}^{r} \text{ fixed}} - (\mathbf{p}^{s*} - c\mathbf{1})^{\top} \left[M^{ss}\frac{\partial \mathbf{p}^{s*}}{\partial \beta} + M^{sr}\frac{\partial \mathbf{p}^{r*}}{\partial \beta}\right].\end{aligned}$$

Here, the fixed-price demand derivative is evaluated at the equilibrium prices. The own-price terms cancel because M^{ss} is symmetric and $\mathbf{x}^{s*} = M^{ss}(\mathbf{p}^{s*} - c\mathbf{1})$. Using $M^{sr} = -\beta M^{ss} K^{r}$ therefore yields

$$\frac{\partial \pi^{s*}}{\partial \beta} = (\mathbf{p}^{s*} - c\mathbf{1})^{\top} \left.\frac{\partial \mathbf{x}^{s}}{\partial \beta}\right|_{\mathbf{p}^{s},\mathbf{p}^{r} \text{ fixed}} + \beta (\mathbf{x}^{s*})^{\top} K^{r} \frac{\partial \mathbf{p}^{r*}}{\partial \beta}.$$

At fixed prices, differentiating the consumer equations and using $M = Q^{-1}$ gives $\left.\frac{\partial \mathbf{x}^{s}}{\partial \beta}\right|_{\mathbf{p}^{s},\mathbf{p}^{r} \text{ fixed}} = -M^{ss}\mathbf{x}^{r*} - M^{sr}\mathbf{x}^{s*}$. Differentiating firm r's equilibrium best response in (A.3) with respect to β gives

$$\frac{\partial \mathbf{p}^{r*}}{\partial \beta} = -\frac{1}{2}K^{s}(a\mathbf{1} - \mathbf{p}^{s*}) + \frac{\beta}{2}K^{s}\frac{\partial \mathbf{p}^{s*}}{\partial \beta} = -\frac{1}{2}\mathbf{x}^{s*} - \frac{\beta}{2}K^{s}\mathbf{x}^{r*} + \frac{\beta}{2}K^{s}\frac{\partial \mathbf{p}^{s*}}{\partial \beta} \leq -\frac{1}{2}\mathbf{x}^{s*}.$$

The second equality uses $Q^{s}\mathbf{x}^{s*} + \beta \mathbf{x}^{r*} = a\mathbf{1} - \mathbf{p}^{s*}$, and the inequality follows from $K^{s} \geq 0$ and $\frac{\partial \mathbf{p}^{s*}}{\partial \beta} < 0$ which is a result in Proposition 2. Substituting the fixed-price demand derivative into the profit derivative above and using $M^{sr} = -\beta M^{ss} K^{r}$ and $(\mathbf{p}^{s*} - c\mathbf{1})^{\top} M^{ss} = (\mathbf{x}^{s*})^{\top}$ gives

$$\frac{\partial \pi^{s*}}{\partial \beta} = (\mathbf{x}^{s*})^{\top} \left[-\mathbf{x}^{r*} + \beta K^{r}\left(\mathbf{x}^{s*} + \frac{\partial \mathbf{p}^{r*}}{\partial \beta}\right)\right] \leq -(\mathbf{x}^{s*})^{\top}\left(\mathbf{x}^{r*} - \frac{\beta}{2}K^{r}\mathbf{x}^{s*}\right).$$

It remains to show that $\mathbf{x}^{r*} - \frac{\beta}{2}K^{r}\mathbf{x}^{s*} > \mathbf{0}$. Multiplying (A.8) by $2(I_{2n} - W)$, using $(I_{2n} - W)\widehat{\mathbf{b}} = \mathbf{1}$ and $QM = I_{2n}$, gives $2Q^{r}\mathbf{x}^{r*} - \beta \mathbf{x}^{s*} = (a - c)\left[\mathbf{1} - \beta(M^{ss} + M^{sr})\mathbf{1}\right]$. Since $K^{r} = (Q^{r})^{-1}$, it follows that $\mathbf{x}^{r*} - \frac{\beta}{2}K^{r}\mathbf{x}^{s*} = \frac{a-c}{2}K^{r}\left[\mathbf{1} - \beta(M^{ss} + M^{sr})\mathbf{1}\right]$. Furthermore, $QM\mathbf{1} = \mathbf{1}$ implies $(M^{ss} + M^{sr})\mathbf{1} = K^{s}\left[\mathbf{1} - \beta(M^{rr} + M^{rs})\mathbf{1}\right] \leq K^{s}\mathbf{1}$, where the inequality follows from $K^{s} \geq 0$ and (A.9). Using the bound on $K^{s}\mathbf{1}$ established earlier, we obtain

$$\mathbf{1} - \beta(M^{ss} + M^{sr})\mathbf{1} \geq \mathbf{1} - \beta K^{s}\mathbf{1} \geq \left(1 - \frac{\beta}{1 - \delta_{s} d^{s}_{\max}}\right)\mathbf{1} > \mathbf{0},$$

where the last inequality follows from Assumption 2. As $a > c$ and K^{r} is nonnegative with strictly positive

diagonal entries, this establishes $\mathbf{x}^{r*} - \frac{\beta}{2} K^r \mathbf{x}^{s*} > \mathbf{0}$. Together with $\mathbf{x}^{s*} > \mathbf{0}$, the preceding profit bound yields $\frac{\partial \pi^{s*}}{\partial \beta} < 0$.

Finally, we examine the effect of an increase in δ_s on equilibrium profits. Consider a change in δ_A; the argument for δ_B is symmetric.

At fixed prices, differentiating the demand system in Proposition 1 and using (A.7) gives

$$\frac{\partial}{\partial \delta_A} \begin{pmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{pmatrix} \Bigg|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} = M \operatorname{diag}(G^A, 0) M \begin{pmatrix} a\mathbf{1} - \mathbf{p}^A \\ a\mathbf{1} - \mathbf{p}^B \end{pmatrix} = M \begin{pmatrix} G^A \mathbf{x}^A \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} M^{AA} G^A \mathbf{x}^A \\ M^{BA} G^A \mathbf{x}^A \end{pmatrix}. \tag{A.11}$$

For product A, $\mathbf{x}^{A*} = M^{AA}(a\mathbf{1} - \mathbf{p}^{A*}) + M^{AB}(a\mathbf{1} - \mathbf{p}^{B*})$. Differentiating with respect to δ_A, holding all other primitives fixed, gives

$$\begin{aligned} \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} &= \frac{\partial M^{AA}}{\partial \delta_A}(a\mathbf{1} - \mathbf{p}^{A*}) + \frac{\partial M^{AB}}{\partial \delta_A}(a\mathbf{1} - \mathbf{p}^{B*}) - M^{AA} \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} - M^{AB} \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A} \\ &= \frac{\partial \mathbf{x}^A}{\partial \delta_A} \Bigg|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} - M^{AA} \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} - M^{AB} \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A}. \end{aligned}$$

The last equality collects the first two terms into the fixed-price demand derivative, evaluated at equilibrium prices. Similarly, for product B, $\frac{\partial \mathbf{x}^{B*}}{\partial \delta_A} = \frac{\partial \mathbf{x}^B}{\partial \delta_A}\Big|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} - M^{BA} \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} - M^{BB} \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A}$.

For firm A, applying the product rule and substituting the demand derivative gives

$$\begin{aligned} \frac{\partial \pi^{A*}}{\partial \delta_A} &= \left(\frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} \right)^\top \mathbf{x}^{A*} + (\mathbf{p}^{A*} - c\mathbf{1})^\top \frac{\partial \mathbf{x}^A}{\partial \delta_A} \Bigg|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} - (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AA} \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} - (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AB} \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A} \\ &= (\mathbf{p}^{A*} - c\mathbf{1})^\top \frac{\partial \mathbf{x}^A}{\partial \delta_A} \Bigg|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} - (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AB} \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A}. \end{aligned}$$

In the last equality, the own-price terms cancel by $\mathbf{x}^{A*} = M^{AA}(\mathbf{p}^{A*} - c\mathbf{1})$ and the symmetry of M^{AA}. Similarly, using $\mathbf{x}^{B*} = M^{BB}(\mathbf{p}^{B*} - c\mathbf{1})$ and the symmetry of M^{BB} gives

$$\frac{\partial \pi^{B*}}{\partial \delta_A} = (\mathbf{p}^{B*} - c\mathbf{1})^\top \frac{\partial \mathbf{x}^B}{\partial \delta_A} \Bigg|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} - (\mathbf{p}^{B*} - c\mathbf{1})^\top M^{BA} \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A}.$$

For firm A, substitute the fixed-price demand derivative (A.11) $\frac{\partial \mathbf{x}^A}{\partial \delta_A}\Big|_{\mathbf{p}^A, \mathbf{p}^B \text{ fixed}} = M^{AA} G^A \mathbf{x}^{A*}$ and use

$M^{AB} = -\beta M^{AA} K^B, (\mathbf{x}^{A*})^\top = (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AA}$ yields

$$\begin{aligned}\frac{\partial \pi^{A*}}{\partial \delta_A} &= (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AA} G^A \mathbf{x}^{A*} + \beta (\mathbf{p}^{A*} - c\mathbf{1})^\top M^{AA} K^B \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A} \\ &= \underbrace{(\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*}}_{\text{direct demand benefit}} + \underbrace{\beta (\mathbf{x}^{A*})^\top K^B \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A}}_{\substack{\text{loss from the rival's} \\ \text{price response}}}.\end{aligned}$$

The nonnegative direct demand effect and the nonpositive rival-price effect make the firm's own profit response ambiguous. To illustrate, consider two consumers connected by a unit-weight link on each network. Fix $\beta = \frac{1}{2}$ and $\delta_B = \frac{15}{32}$, and let $0 < \delta_A < \frac{15}{32}$. The equilibrium conditions give

$$\frac{\partial \pi^{A*}}{\partial \delta_A} = \frac{51(a-c)^2(5-9\delta_A)(5-17\delta_A)(25-51\delta_A)}{2(15-17\delta_A)^3(9-17\delta_A)^2}.$$

This derivative is positive for $0 < \delta_A < \frac{5}{17}$ and negative for $\frac{5}{17} < \delta_A < \frac{15}{32}$. However at $\beta = 0$, the latter vanishes, so own profit strictly increases whenever $G^A \neq 0$.

Similarly, for firm B, we obtain

$$\frac{\partial \pi^{B*}}{\partial \delta_A} = \underbrace{-\beta (\mathbf{x}^{B*})^\top K^A G^A \mathbf{x}^{A*}}_{\text{loss from demand diversion}} + \underbrace{\beta (\mathbf{x}^{B*})^\top K^A \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A}}_{\substack{\text{loss from the rival's} \\ \text{price response}}} \leq 0.$$

Both terms are nonpositive as equilibrium consumption is positive and $\frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} \leq \mathbf{0}$. The rival's profit strictly decreases if $\beta > 0$ and $G^A \neq 0$, because the positive diagonal entries of K^A make the first term strictly negative. At $\beta = 0$, the rival's profit is unaffected.

Proof of part (ii). As in the proof of Proposition 2, we consider $G^A(t) = G^A + t(G^{A\prime} - G^A), t \in [0,1]$, for $G^{A\prime} \geq G^A$ and holding G^B fixed. To examine the rival's profit, differentiate the consumer demand system along the network path. Using $\frac{\mathrm{d}M}{\mathrm{d}t} = \delta_A M \operatorname{diag}(G^{A\prime} - G^A, 0) M$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{x}^{A*} \\ \mathbf{x}^{B*}\end{pmatrix} = \frac{\mathrm{d}M}{\mathrm{d}t}\begin{pmatrix}a\mathbf{1} - \mathbf{p}^{A*} \\ a\mathbf{1} - \mathbf{p}^{B*}\end{pmatrix} - M\begin{pmatrix}\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t} \\ \frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}t}\end{pmatrix} = \delta_A M\begin{pmatrix}(G^{A\prime} - G^A)\mathbf{x}^{A*} \\ \mathbf{0}\end{pmatrix} - M\begin{pmatrix}\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t} \\ \frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}t}\end{pmatrix}.$$

Its B-block gives $\frac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}t} = \delta_A M^{BA}(G^{A\prime} - G^A)\mathbf{x}^{A*} - M^{BA}\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t} - M^{BB}\frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}t}$. Differentiating $\pi^{B*} = (\mathbf{p}^{B*} -$

$c\mathbf{1})^\top \mathbf{x}^{B*}$ and using the symmetry of M^{BB} therefore yields

$$\frac{\mathrm{d}\pi^{B*}}{\mathrm{d}t} = (\mathbf{p}^{B*} - c\mathbf{1})^\top M^{BA}\left[\delta_A(G^{A\prime} - G^A)\mathbf{x}^{A*} - \frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t}\right] + \left(\frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}t}\right)^\top \left[\mathbf{x}^{B*} - M^{BB}(\mathbf{p}^{B*} - c\mathbf{1})\right].$$

The last term vanishes by noting $\mathbf{x}^{B*} = M^{BB}(\mathbf{p}^{B*} - c\mathbf{1})$. Using this identity and $M^{BA} = -\beta M^{BB} K^A$ gives

$$\frac{\mathrm{d}\pi^{B*}}{\mathrm{d}t} = -\beta\,\delta_A(\mathbf{x}^{B*})^\top K^A(G^{A\prime} - G^A)\mathbf{x}^{A*} + \beta(\mathbf{x}^{B*})^\top K^A \frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t} \le 0.$$

Both terms are nonpositive because $G^{A\prime} - G^A \ge 0$, $\mathbf{x}^{B*} > 0$, and $\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}t} \le \mathbf{0}$. Integrating over $t \in [0,1]$ and applying the same argument with the firm labels reversed establishes $\pi^{r*}(G^{s\prime}, G^r) \le \pi^{r*}(G^s, G^r)$.

If $\delta_A > 0$, $\beta > 0$, and $G^{A\prime} \neq G^A$, then $(G^{A\prime} - G^A)\mathbf{x}^{A*}$ is nonnegative and nonzero. Together with $K^A \ge I_n$ and $\mathbf{x}^{B*} > \mathbf{0}$, this makes the first term strictly negative, while the second is nonpositive. Hence $\frac{\mathrm{d}\pi^{B*}}{\mathrm{d}t} < 0$ along the path, and integration implies a strict decrease in firm B's equilibrium profit.

To illustrate the ambiguity, consider two consumers with $a = 1$, $c = 0$, $\delta_A = \frac{2}{9}$, $\delta_B = \frac{2}{7}$, and $\beta = \frac{2}{3}$. Initially, network A has no links, while network B connects the two consumers with a unit-weight edge. Network A then gains an edge of weight $w \in (0,1]$. Firm A's initial equilibrium profit is $\frac{1125}{12274}$. Adding an edge of weight $w = \frac{1}{2}$ raises its profit to $\frac{5}{54}$, whereas adding a unit-weight edge lowers it to $\frac{5}{56}$. Hence, network expansion can either increase or decrease the expanding firm's profit.

It remains to establish the own-profit comparison for sufficiently weak substitution. Fix the initial and expanded networks and all scalar primitives other than β. If $\delta_s(G^{s\prime} - G^s) = 0$, the two economies coincide and $\pi^{s*}(G^{s\prime}, G^r; \beta) - \pi^{s*}(G^s, G^r; \beta) = 0$. Otherwise, let $K^s = (I - \delta_s G^s)^{-1}$ and $K^{s\prime} = (I - \delta_s G^{s\prime})^{-1}$. At $\beta = 0$, the equilibrium price is $\frac{a+c}{2}$ for every consumer. The resolvent identity therefore gives

$$\begin{aligned}\pi^{s*}(G^{s\prime}, G^r; 0) - \pi^{s*}(G^s, G^r; 0) &= \frac{(a-c)^2}{4}\mathbf{1}^\top(K^{s\prime} - K^s)\mathbf{1} \\ &= \frac{(a-c)^2\delta_s}{4}\mathbf{1}^\top K^{s\prime}(G^{s\prime} - G^s)K^s\mathbf{1} > 0.\end{aligned}$$

The strict inequality follows because $K^s \ge I$ and $K^{s\prime} \ge I$ componentwise, while $\delta_s(G^{s\prime} - G^s)$ is nonnegative and nonzero. Let $q = 1 - \max\{\delta_s d^{s\prime}_{\max}, \delta_r d^r_{\max}\} > 0$. Both compared economies satisfy Assumption 2 for $0 \le \beta < q$. By Theorem 1, the difference $\pi^{s*}(G^{s\prime}, G^r; \beta) - \pi^{s*}(G^s, G^r; \beta)$ is continuous at $\beta = 0$. Hence

there exists $\bar{\beta} \in (0,q)$ such that

$$\pi^{s*}(G^{s\prime}, G^r; \beta) - \pi^{s*}(G^s, G^r; \beta) > 0 \qquad \text{for every } 0 \leq \beta < \bar{\beta}.$$

This establishes the claimed weak-substitution comparison.

□

Proof of Lemma 1. Assumption 2 ensures that equilibrium consumption is strictly positive. The consumer first-order conditions imply $a - p_i^{s*} = x_i^{s*} + \beta x_i^{r*} - \delta_s \sum_{j \in N} g_{ij}^s x_j^{s*}$ for $r \neq s$. Substituting into (1) gives

$$u_i^* = \frac{1}{2}(x_i^{A*})^2 + \frac{1}{2}(x_i^{B*})^2 + \beta x_i^{A*} x_i^{B*}.$$

Summing over consumers yields $CS^* = \frac{1}{2}\|\mathbf{x}^{A*}\|_2^2 + \frac{1}{2}\|\mathbf{x}^{B*}\|_2^2 + \beta(\mathbf{x}^{A*})^\top \mathbf{x}^{B*}$, which establishes the consumer-surplus expression in (6).

To derive the social welfare expression, sum (1) over consumers and add the firms' profits cancels payments $\sum_{s\in\{A,B\}}(\mathbf{p}^{s*})^\top \mathbf{x}^{s*}$, yields the expression for SW^* in (6). □

Lemma 2. *Holding the two networks and all other parameters fixed, the equilibrium welfare derivatives with respect to δ_A are*

$$\begin{cases} \dfrac{\partial CS^*}{\partial \delta_A} = \left(\mathbf{x}^{A*} + \beta \mathbf{x}^{B*}\right)^\top \dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \left(\mathbf{x}^{B*} + \beta \mathbf{x}^{A*}\right)^\top \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A}, \\ \dfrac{\partial SW^*}{\partial \delta_A} = (\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*} + \displaystyle\sum_{s\in\{A,B\}} (\mathbf{p}^{s*} - c\mathbf{1} + \delta_s G^s \mathbf{x}^{s*})^\top \dfrac{\partial \mathbf{x}^{s*}}{\partial \delta_A}. \end{cases} \tag{A.12}$$

The corresponding expressions for δ_B follow by interchanging the labels.

Proof of Lemma 2. By Lemma 1, $CS^* = \frac{1}{2}\|\mathbf{x}^{A*}\|_2^2 + \frac{1}{2}\|\mathbf{x}^{B*}\|_2^2 + \beta(\mathbf{x}^{A*})^\top \mathbf{x}^{B*}$. Holding the two networks and all other parameters fixed, differentiating with respect to δ_A gives

$$\frac{\partial CS^*}{\partial \delta_A} = \left(\mathbf{x}^{A*} + \beta \mathbf{x}^{B*}\right)^\top \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \left(\mathbf{x}^{B*} + \beta \mathbf{x}^{A*}\right)^\top \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A}.$$

Next, using $SW^* = CS^* + \sum_{s\in\{A,B\}} (\mathbf{p}^{s*} - c\mathbf{1})^\top \mathbf{x}^{s*}$ and applying the product rule yields

$$\frac{\partial SW^*}{\partial \delta_A} = \frac{\partial CS^*}{\partial \delta_A} + \sum_{s\in\{A,B\}} \left[(\mathbf{p}^{s*} - c\mathbf{1})^\top \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A} + (\mathbf{x}^{s*})^\top \frac{\partial \mathbf{p}^{s*}}{\partial \delta_A} \right]. \tag{A.13}$$

To eliminate the price derivatives, consider the consumer-equilibrium conditions:

$$\begin{cases} (I - \delta_A G^A)\mathbf{x}^{A*} + \beta \mathbf{x}^{B*} = a\mathbf{1} - \mathbf{p}^{A*}, \\ (I - \delta_B G^B)\mathbf{x}^{B*} + \beta \mathbf{x}^{A*} = a\mathbf{1} - \mathbf{p}^{B*}. \end{cases}$$

Differentiating with respect to δ_A, including the dependence of both prices and consumption on this parameter, gives

$$\begin{cases} (I - \delta_A G^A) \dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \beta \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} = G^A \mathbf{x}^{A*} - \dfrac{\partial \mathbf{p}^{A*}}{\partial \delta_A}, \\ (I - \delta_B G^B) \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} + \beta \dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} = -\dfrac{\partial \mathbf{p}^{B*}}{\partial \delta_A}. \end{cases} \tag{A.14}$$

Premultiplying the first equation in (A.14) by $(\mathbf{x}^{A*})^\top$, premultiplying the second by $(\mathbf{x}^{B*})^\top$, and rearranging their sum gives

$$\begin{aligned} \sum_{s\in\{A,B\}} (\mathbf{x}^{s*})^\top \frac{\partial \mathbf{p}^{s*}}{\partial \delta_A} &= (\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*} + \sum_{s\in\{A,B\}} \delta_s (\mathbf{x}^{s*})^\top G^s \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A} - \sum_{s\in\{A,B\}} (\mathbf{x}^{s*})^\top \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A} \\ &\quad - \beta \left[(\mathbf{x}^{A*})^\top \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A} + (\mathbf{x}^{B*})^\top \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} \right] \\ &= -\frac{\partial CS^*}{\partial \delta_A} + (\mathbf{x}^{A*})^\top G^A \mathbf{x}^{A*} + \sum_{s\in\{A,B\}} \delta_s (\mathbf{x}^{s*})^\top G^s \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A}. \end{aligned}$$

Substituting this identity into (A.13) cancels the positive and negative $\frac{\partial CS^*}{\partial \delta_A}$ terms. Finally, symmetry of G^s implies $(\mathbf{x}^{s*})^\top G^s \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A} = (G^s \mathbf{x}^{s*})^\top \frac{\partial \mathbf{x}^{s*}}{\partial \delta_A}$, which establishes the social-welfare derivative in (A.12). □

Proof of Proposition 4. Proof of part (i). When $\beta = 0$, the consumer network of two products decouple. Thus, we have $\mathbf{p}^{s*} = \frac{a+c}{2}\mathbf{1}$ and $\mathbf{x}^{s*} = \frac{a-c}{2} K^s \mathbf{1}$ by Corollary 1. Differentiating the consumer-equilibrium equation for product A with respect to δ_A gives $(I - \delta_A G^A) \frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} = G^A \mathbf{x}^{A*}$. Hence,

$$\begin{cases} \dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} = K^A G^A \mathbf{x}^{A*} \geq 0, \\ \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} = 0. \end{cases}$$

If $G^A \neq 0$, the first vector has at least one strictly positive component because $\mathbf{x}^{A*} > 0$ and K^A is nonnegative with a strictly positive diagonal. Both welfare derivatives in (A.12) are therefore strictly positive. If $G^A = 0$, changing δ_A has no effect.

For the finite network comparison, let $\widetilde{\mathbf{x}}^{A*}$ denote equilibrium consumption under $G^{A\prime}$. The initial and final consumer-equilibrium equations are

$$\begin{cases} (I - \delta_A G^A)\mathbf{x}^{A*} = \dfrac{a-c}{2}\mathbf{1}, \\ (I - \delta_A G^{A\prime})\widetilde{\mathbf{x}}^{A*} = \dfrac{a-c}{2}\mathbf{1}. \end{cases}$$

Subtracting and rearranging yields $\widetilde{\mathbf{x}}^{A*} - \mathbf{x}^{A*} = \delta_A (I - \delta_A G^{A\prime})^{-1}(G^{A\prime} - G^A)\mathbf{x}^{A*} \geq 0$. This vector is nonzero if $\delta_A > 0$ and $G^{A\prime} \neq G^A$. As $\beta = 0$ and consumption of product B is unchanged, it follows from (6) that

$$CS^*(G^{A\prime}, G^B) - CS^*(G^A, G^B) = \frac{1}{2}\left(\|\widetilde{\mathbf{x}}^{A*}\|_2^2 - \|\mathbf{x}^{A*}\|_2^2\right) \geq 0.$$

Equilibrium prices remain $\frac{a+c}{2}\mathbf{1}$. The change in total profit is therefore $\frac{a-c}{2}\mathbf{1}^\top\left(\widetilde{\mathbf{x}}^{A*} - \mathbf{x}^{A*}\right) \geq 0$. This proves the social-welfare comparison. Both welfare inequalities are strict if $\delta_A > 0$ and $G^{A\prime} \neq G^A$. The argument for B is identical.

Proof of part (ii). We first consider the symmetric reference configuration $\delta_A G^A = \delta_B G^B = D$. Symmetry of the two effective networks and uniqueness of equilibrium imply $\mathbf{x}^{A*} = \mathbf{x}^{B*} = \mathbf{q} > 0$ and $\mathbf{p}^{A*} = \mathbf{p}^{B*}$. Moreover, $\mathbf{p}^{A*} - c\mathbf{1} = \mathbf{p}^{B*} - c\mathbf{1} > 0$. We differentiate the equilibrium conditions before evaluating the derivatives at this configuration. A unilateral increase in δ_A need not preserve symmetry, so the two consumption derivatives need not be equal.

Evaluating (A.14) at the reference configuration gives

$$\begin{cases} (I - D)\dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \beta \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} = G^A\mathbf{q} - \dfrac{\partial \mathbf{p}^{A*}}{\partial \delta_A}, \\ (I - D)\dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} + \beta \dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} = -\dfrac{\partial \mathbf{p}^{B*}}{\partial \delta_A}. \end{cases}$$

Adding these equations yields

$$\left[(1+\beta)I - D\right]\left(\frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A}\right) = G^A\mathbf{q} - \frac{\partial \mathbf{p}^{A*}}{\partial \delta_A} - \frac{\partial \mathbf{p}^{B*}}{\partial \delta_A}.$$

Both price derivatives are componentwise nonpositive by Proposition 2. Moreover, Assumption 2 ensures that $\left[(1+\beta)I - D\right]^{-1}$ is componentwise nonnegative with a strictly positive diagonal. Consequently, $\frac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \frac{\partial \mathbf{x}^{B*}}{\partial \delta_A} \geq 0$, with at least one strictly positive component if $G^A \neq 0$.

Substituting $\mathbf{x}^{A*} = \mathbf{x}^{B*} = \mathbf{q}$ and $\mathbf{p}^{A*} = \mathbf{p}^{B*}$ into (A.12) gives

$$\begin{cases} \dfrac{\partial CS^*}{\partial \delta_A} = (1+\beta)\mathbf{q}^\top \left(\dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} \right) > 0, \\ \dfrac{\partial SW^*}{\partial \delta_A} = \mathbf{q}^\top G^A \mathbf{q} + \left(\mathbf{p}^{A*} - c\mathbf{1} + D\mathbf{q}\right)^\top \left(\dfrac{\partial \mathbf{x}^{A*}}{\partial \delta_A} + \dfrac{\partial \mathbf{x}^{B*}}{\partial \delta_A} \right) > 0, \end{cases}$$

whenever $G^A \neq 0$. If $G^A = 0$, both derivatives are zero. The argument for δ_B follows by interchanging the labels.

We next extend these derivative conclusions to nearby asymmetric configurations. Write $D^s = \delta_s G^s$ and regard CS^* and SW^* as functions of the independent off-diagonal entries of D^A and D^B. For $i < j$, let $H^{ij} = \mathbf{e}_i \mathbf{e}_j^\top + \mathbf{e}_j \mathbf{e}_i^\top$, where $\mathbf{e}_i$ is the ith standard basis vector. Thus, H^{ij} has ones in positions (i,j) and (j,i) and zeros elsewhere. For small $\tau > 0$, the perturbation $D^s + \tau H^{ij}$ increases the effective weight of the undirected edge $\{i,j\}$ by τ, leaving all other edge weights unchanged and preserving symmetry.

Fix $i < j$ and consider the perturbation $D^A(\tau) = D + \tau H^{ij}$, holding $D^B(\tau) = D$ and all other primitives fixed. Let $\mathbf{x}^{s*}(\tau)$ and $\mathbf{p}^{s*}(\tau)$ denote the resulting equilibrium consumption and price vectors. The consumer-equilibrium conditions are

$$\begin{cases} (I - D - \tau H^{ij})\mathbf{x}^{A*}(\tau) + \beta \mathbf{x}^{B*}(\tau) = a\mathbf{1} - \mathbf{p}^{A*}(\tau), \\ (I - D)\mathbf{x}^{B*}(\tau) + \beta \mathbf{x}^{A*}(\tau) = a\mathbf{1} - \mathbf{p}^{B*}(\tau). \end{cases}$$

At $\tau = 0$, symmetry implies $\mathbf{x}^{A*}(0) = \mathbf{x}^{B*}(0) = \mathbf{q} > 0$ and $\mathbf{p}^{A*}(0) = \mathbf{p}^{B*}(0) = \mathbf{p}$, with $\mathbf{p} - c\mathbf{1} > 0$. All derivatives in the following calculations are evaluated at $\tau = 0$.

Differentiating the equilibrium conditions and applying the product rule gives

$$\begin{cases} -H^{ij}\mathbf{q} + (I - D)\dfrac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau} + \beta \dfrac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau} = -\dfrac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}\tau}, \\ (I - D)\dfrac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau} + \beta \dfrac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau} = -\dfrac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}\tau}. \end{cases}$$

Adding these equations and rearranging yields

$$\frac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau}+\frac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau}=\left[(1+\beta)I-D\right]^{-1}\left(H^{ij}\mathbf{q}-\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}\tau}-\frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}\tau}\right).$$

The price-monotonicity argument in Proposition 2 applies to this nonnegative perturbation of the effective network, giving $\frac{\mathrm{d}\mathbf{p}^{A*}}{\mathrm{d}\tau}\leq 0$ and $\frac{\mathrm{d}\mathbf{p}^{B*}}{\mathrm{d}\tau}\leq 0$. Moreover, $H^{ij}\mathbf{q}=q_j\mathbf{e}_i+q_i\mathbf{e}_j$ is nonnegative and nonzero. As $\left[(1+\beta)I-D\right]^{-1}$ is componentwise nonnegative with a strictly positive diagonal, $\frac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau}+\frac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau}\geq 0$, with at least one strictly positive component.

The derivatives along this perturbation at $\tau=0$ are precisely the partial derivatives with respect to the independent edge coordinate D^A_{ij} at (D,D). As $\frac{\mathrm{d}D^A(\tau)}{\mathrm{d}\tau}=H^{ij}$, the welfare decomposition in (A.12) gives

$$\frac{\partial CS^*}{\partial D^A_{ij}}=(1+\beta)\mathbf{q}^\top\left(\frac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau}+\frac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau}\right)>0,$$
$$\frac{\partial SW^*}{\partial D^A_{ij}}=\mathbf{q}^\top H^{ij}\mathbf{q}+(\mathbf{p}-c\mathbf{1}+D\mathbf{q})^\top\left(\frac{\mathrm{d}\mathbf{x}^{A*}}{\mathrm{d}\tau}+\frac{\mathrm{d}\mathbf{x}^{B*}}{\mathrm{d}\tau}\right)>0.$$

Here the sum of the consumption derivatives is nonnegative and nonzero, $\mathbf{q}>0$, $\mathbf{p}-c\mathbf{1}+D\mathbf{q}>0$, and $\mathbf{q}^\top H^{ij}\mathbf{q}=2q_iq_j>0$. Interchanging the labels establishes $\frac{\partial CS^*}{\partial D^B_{ij}}>0$ and $\frac{\partial SW^*}{\partial D^B_{ij}}>0$.

The preceding one-edge perturbations establish strict positivity of all relevant partial derivatives at the symmetric reference point (D,D). Each of these derivatives is a continuous function of both effective networks under Assumption 2. Because there are finitely many such derivatives, there exists $\varepsilon_D>0$ such that $\frac{\partial CS^*}{\partial D^s_{ij}}>0$ and $\frac{\partial SW^*}{\partial D^s_{ij}}>0$ for every $s\in\{A,B\}$ and $i<j$, whenever $\max_{s\in\{A,B\}}\|D^s-D\|_\infty<\varepsilon_D$. This is a joint neighborhood of (D,D), allowing both effective networks to differ from D simultaneously. The chain rule therefore gives $\frac{\partial CS^*}{\partial \delta_s}=\sum_{i<j}G^s_{ij}\frac{\partial CS^*}{\partial D^s_{ij}}\geq 0$ and $\frac{\partial SW^*}{\partial \delta_s}=\sum_{i<j}G^s_{ij}\frac{\partial SW^*}{\partial D^s_{ij}}\geq 0$, where all derivatives are evaluated at the actual configuration (D^A,D^B).

Now we compare two symmetric reference configurations with common effective networks $\delta^AG^{A\prime}=\delta^BG^{B\prime}=D'\geq D=\delta^AG^A=\delta^BG^B$. Let $\mathbf{q},\mathbf{p}$ denote the common consumption and price vectors at the initial reference equilibrium, and let $\mathbf{q}',\mathbf{p}'$ denote their counterparts at the final reference equilibrium. The

consumer-equilibrium conditions are

$$\begin{cases} \big[(1+\beta)I - D\big]\mathbf{q} = a\mathbf{1} - \mathbf{p}, \\ \big[(1+\beta)I - D'\big]\mathbf{q}' = a\mathbf{1} - \mathbf{p}'. \end{cases} \tag{A.15}$$

The price monotonicity result in Proposition 2 implies $\mathbf{p}' \leq \mathbf{p}$. Subtracting the two equilibrium equations gives $\mathbf{q}' - \mathbf{q} = \big[(1+\beta)I - D'\big]^{-1}\big[(D'-D)\mathbf{q} + \mathbf{p} - \mathbf{p}'\big] \geq 0$. If $D' \neq D$, at least one component is strictly positive. As consumer surplus at a symmetric equilibrium is $(1+\beta)\|\mathbf{q}\|_2^2$, we obtain $CS^{*\prime} - CS^* = (1+\beta)\big(\|\mathbf{q}'\|_2^2 - \|\mathbf{q}\|_2^2\big) \geq 0$, with strict inequality when $D' \neq D$.

For social welfare, (6) gives $SW^* = 2(a-c)\mathbf{1}^\top\mathbf{q} - (1+\beta)\|\mathbf{q}\|_2^2 + 2\mathbf{q}^\top D\mathbf{q}$, with an analogous expression at the final reference equilibrium. The two equations in (A.15) imply $2(a-c)\mathbf{1} = (1+\beta)(\mathbf{q}+\mathbf{q}') - D\mathbf{q} - D'\mathbf{q}' + \mathbf{p} + \mathbf{p}' - 2c\mathbf{1}$. In addition, symmetry of D and D' gives $(\mathbf{q}')^\top D'\mathbf{q}' - \mathbf{q}^\top D\mathbf{q} = (D'\mathbf{q}' + D\mathbf{q})^\top(\mathbf{q}' - \mathbf{q}) + \mathbf{q}^\top(D'-D)\mathbf{q}'$. Using these identities yields

$$\begin{aligned} SW^{*\prime} - SW^* &= \big[2(a-c)\mathbf{1} - (1+\beta)(\mathbf{q}+\mathbf{q}')\big]^\top(\mathbf{q}' - \mathbf{q}) + 2\Big[(\mathbf{q}')^\top D'\mathbf{q}' - \mathbf{q}^\top D\mathbf{q}\Big] \\ &= \big(\mathbf{p} + \mathbf{p}' - 2c\mathbf{1} + D\mathbf{q} + D'\mathbf{q}'\big)^\top(\mathbf{q}' - \mathbf{q}) + 2\mathbf{q}^\top(D'-D)\mathbf{q}' \geq 0. \end{aligned}$$

Both terms are nonnegative because $\mathbf{p} - c\mathbf{1} > 0$, $\mathbf{p}' - c\mathbf{1} > 0$, $\mathbf{q}' - \mathbf{q} \geq 0$, and $D' \geq D \geq 0$. If $D' \neq D$, the second term is strictly positive because $\mathbf{q}, \mathbf{q}' > 0$.

When $D' \neq D$, both welfare differences are therefore strictly positive at the reference configurations. By continuity of equilibrium welfare, there exists $\varepsilon_{D,D'} > 0$ such that these strict inequalities persist whenever the initial and final effective networks satisfy the stated bounds around D and D'.

It remains to consider $D' = D$. Choose $\varepsilon_{D,D} = \varepsilon_D$ and consider $D^s(t) = \delta_s[(1-t)G^s + tG^{s\prime}]$, $t \in [0,1]$. The endpoint bounds imply $\max_{s\in\{A,B\}} \|D^s(t) - D\|_\infty < \varepsilon_D$ throughout the path, so all effective-edge partial derivatives of both welfare measures remain strictly positive. Along this path, the chain rule gives

$$\begin{aligned} \frac{\mathrm{d}CS^*}{\mathrm{d}t} &= \sum_{s\in\{A,B\}} \sum_{i<j} \delta_s\big(G^{s\prime}_{ij} - G^s_{ij}\big)\frac{\partial CS^*}{\partial D^s_{ij}} \geq 0, \\ \frac{\mathrm{d}SW^*}{\mathrm{d}t} &= \sum_{s\in\{A,B\}} \sum_{i<j} \delta_s\big(G^{s\prime}_{ij} - G^s_{ij}\big)\frac{\partial SW^*}{\partial D^s_{ij}} \geq 0. \end{aligned}$$

Integrating over $t \in [0,1]$ establishes both finite welfare comparisons and completes the proof of part (*ii*).

Proof of part (iii). The strict positive effects established above persist under sufficiently small admissible perturbations, including perturbations that make the effective networks asymmetric. Example 2 establishes that increasing a network-effect parameter or adding links can instead reduce both welfare measures in asymmetric networks. Thus, neither comparative-static sign is determined in general. □

Proof of Proposition 5. Stack the two price vectors and consumption vectors as $\mathbf{p} = ((\mathbf{p}^A)^\top, (\mathbf{p}^B)^\top)^\top$ and $\mathbf{x} = ((\mathbf{x}^A)^\top, (\mathbf{x}^B)^\top)^\top$. Assumption 2 implies Assumption 1. As shown in the proof of Proposition 1, Q is symmetric and positive definite. Hence, $M = Q^{-1}$ is also symmetric and positive definite.

For prices inducing an interior consumer equilibrium, Proposition 1 gives $\mathbf{x} = M(a\mathbf{1}_{2n} - \mathbf{p})$. Substituting this expression into (8) yields $\Pi(\mathbf{p}) = (\mathbf{p} - c\mathbf{1}_{2n})^\top M(a\mathbf{1}_{2n} - \mathbf{p})$. Differentiating with respect to the entire price vector gives $\nabla_{\mathbf{p}}\Pi(\mathbf{p}) = M\big[(a+c)\mathbf{1}_{2n} - 2\mathbf{p}\big] = 0$. As M is invertible, the unique solution is $\tilde{\mathbf{p}} = \frac{a+c}{2}\mathbf{1}_{2n}$. Moreover, the Hessian of this quadratic expression is $-2M \prec 0$, so $\tilde{\mathbf{p}}$ is its unique maximizer. Substitution into consumer demand gives $\tilde{\mathbf{x}} = \frac{a-c}{2}M\mathbf{1}_{2n}$, with

$$\begin{pmatrix} \tilde{\mathbf{x}}^A \\ \tilde{\mathbf{x}}^B \end{pmatrix} = \frac{a-c}{2}\begin{pmatrix} M^{AA} & M^{AB} \\ M^{BA} & M^{BB} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} = \frac{a-c}{2}\begin{pmatrix} (M^{AA}+M^{AB})\mathbf{1} \\ (M^{BA}+M^{BB})\mathbf{1} \end{pmatrix} = \frac{a-c}{2}\begin{pmatrix} M^{AA}(\mathbf{1}-\beta K^B\mathbf{1}) \\ M^{BB}(\mathbf{1}-\beta K^A\mathbf{1}) \end{pmatrix},$$

where the third equality uses $M^{AB} = -\beta M^{AA}K^B$ and $M^{BA} = -\beta M^{BB}K^A$; see (A.1). It follows from $\mathbf{b}^s = K^s\mathbf{1}$ that

$$\begin{cases} \tilde{\mathbf{x}}^A = \dfrac{a-c}{2}M^{AA}\big(\mathbf{1}-\beta\mathbf{b}^B\big), \\ \tilde{\mathbf{x}}^B = \dfrac{a-c}{2}M^{BB}\big(\mathbf{1}-\beta\mathbf{b}^A\big). \end{cases}$$

For each $s \in \{A,B\}$, (A.10) implies $0 \le \beta\mathbf{b}^s < \mathbf{1}$. Proposition 1 establishes that M^{AA} and M^{BB} are componentwise nonnegative and have strictly positive diagonal entries. Hence, $\tilde{\mathbf{x}}^A, \tilde{\mathbf{x}}^B > 0$.

To establish global optimality, we also consider prices that induce zero consumption for some consumers. At any nonnegative consumer equilibrium, the first-order conditions for positive consumption, together with the zero contribution of inactive choices, imply $\mathbf{x}^\top\big(a\mathbf{1}_{2n} - \mathbf{p} - Q\mathbf{x}\big) = 0$. Therefore, the merged firm's profit can always be written as

$$\Pi(\mathbf{p}) = (\mathbf{p} - c\mathbf{1}_{2n})^\top\mathbf{x} = (a-c)\mathbf{1}_{2n}^\top\mathbf{x} - \mathbf{x}^\top Q\mathbf{x} = \frac{(a-c)^2}{4}\mathbf{1}_{2n}^\top Q^{-1}\mathbf{1}_{2n} - (\mathbf{x}-\tilde{\mathbf{x}})^\top Q(\mathbf{x}-\tilde{\mathbf{x}}).$$

Since $Q \succ 0$, this expression is uniquely maximized at $\mathbf{x} = \tilde{\mathbf{x}}$. The preceding argument shows that $\tilde{\mathbf{x}} > 0$ and is implemented by $\tilde{\mathbf{p}}$. Moreover, because all consumption choices are strictly positive, the consumer-equilibrium equations uniquely determine the implementing prices as $\tilde{\mathbf{p}} = a\mathbf{1}_{2n} - Q\tilde{\mathbf{x}}$. This establishes global optimality, uniqueness, and the profit formula.

Finally, subtracting the competitive prices in Theorem 1 from the merger prices gives

$$\begin{cases} \tilde{\mathbf{p}}^A - \mathbf{p}^{A*} = \dfrac{(a-c)\beta}{4}\widehat{\mathbf{b}}^B, \\ \tilde{\mathbf{p}}^B - \mathbf{p}^{B*} = \dfrac{(a-c)\beta}{4}\widehat{\mathbf{b}}^A. \end{cases}$$

Because $\widehat{\mathbf{b}}^A, \widehat{\mathbf{b}}^B > 0$, every price increase is strictly positive when $\beta > 0$.

For the profit comparison, let $\mathbf{p}^*$ denote the stacked competitive price vector. Since the competitive equilibrium is interior, using $M = M^\top$ and $\tilde{\mathbf{p}} = \frac{a+c}{2}\mathbf{1}_{2n}$ gives

$$\begin{aligned} \tilde{\Pi} - (\pi^{A*} + \pi^{B*}) &= \frac{(a-c)^2}{4}\mathbf{1}_{2n}^\top M \mathbf{1}_{2n} - (\mathbf{p}^* - c\mathbf{1}_{2n})^\top M(a\mathbf{1}_{2n} - \mathbf{p}^*) \\ &= (\mathbf{p}^*)^\top M \mathbf{p}^* - (a+c)\mathbf{1}_{2n}^\top M \mathbf{p}^* + \frac{(a+c)^2}{4}\mathbf{1}_{2n}^\top M \mathbf{1}_{2n} \\ &= (\mathbf{p}^* - \tilde{\mathbf{p}})^\top M(\mathbf{p}^* - \tilde{\mathbf{p}}). \end{aligned}$$

Positive definiteness of M makes this difference strictly positive whenever $\beta > 0$. If $\beta = 0$, the competitive and merger prices coincide. Uniqueness of the consumer equilibrium then implies identical consumption and total profit. □

Proof of Proposition 6. Holding prices and other consumers' consumption fixed, define $v_i^s := a - p_i^s + \delta_s \sum_j g_{ij}^s x_j^s$ for $s \in \{A, B\}$. Suppose $\beta > 1$ and consider a choice with $x_i^A > 0$ and $x_i^B > 0$. Without loss of generality, assume $v_i^A \geq v_i^B$. Consider the deviation to $(x_i^A + x_i^B, 0)$, which allocates all consumption to product A while keeping total consumption unchanged. The utilities before and after this deviation are

$$\begin{cases} u_i(x_i^A, x_i^B) = v_i^A x_i^A + v_i^B x_i^B - \dfrac{1}{2}(x_i^A + x_i^B)^2 - (\beta - 1)x_i^A x_i^B, \\ u_i(x_i^A + x_i^B, 0) = v_i^A(x_i^A + x_i^B) - \dfrac{1}{2}(x_i^A + x_i^B)^2. \end{cases}$$

Thus, the utility gain from the deviation is

$$\begin{aligned} u_i(x_i^A + x_i^B, 0) - u_i(x_i^A, x_i^B) &= v_i^A(x_i^A + x_i^B - x_i^A) - v_i^B x_i^B + (\beta - 1)x_i^A x_i^B \\ &= (v_i^A - v_i^B)x_i^B + (\beta - 1)x_i^A x_i^B > 0. \end{aligned}$$

The first term is nonnegative because $v_i^A \geq v_i^B$, and the second is strictly positive because $\beta > 1$ and both initial consumption levels are positive. Hence, consuming both products cannot be a best response. The case $v_i^B > v_i^A$ follows by considering the deviation to $(0, x_i^A + x_i^B)$. Therefore, when $\beta > 1$, every consumer with positive total consumption consumes only one product in any consumption equilibrium.

To derive the no-switch condition, consider a consumer i choosing product A with positive consumption. Optimizing consumption of product A gives $x_i^A = v_i^A > 0$ and utility $\frac{1}{2}(v_i^A)^2$. Switching to product B and reoptimizing consumption yields

$$\max_{x_i^B \geq 0} \left\{ v_i^B x_i^B - \frac{1}{2}(x_i^B)^2 \right\} = \frac{1}{2}\big(\max\{v_i^B, 0\}\big)^2.$$

As $v_i^A > 0$, continuing to consume product A is weakly preferred if and only if $v_i^A \geq v_i^B$. Substituting the definitions of v_i^A and v_i^B, the no-switch condition becomes

$$p_i^A - p_i^B \leq \delta_A \sum_j g_{ij}^A x_j^A - \delta_B \sum_j g_{ij}^B x_j^B.$$

Equality corresponds to indifference between the two products. □

Proof of Example 3. We construct the first equilibrium and verify that neither firm has a profitable deviation.

Let $H := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = G^A = G^B$ denote the common adjacency matrix of the two networks. First, specify consumers' continuation strategies. Whenever $\mathbf{p}^A = (\frac{1}{2}, \frac{1}{2})^\top$ and $p_i^B \geq -1$ for both consumers, select the consumption profile $\mathbf{x}^A = (2,2)^\top$ and $\mathbf{x}^B = \mathbf{0}$. This is a consumer Nash equilibrium: given the other consumer's activity, each consumer's net marginal valuation for product A is $1 - \frac{1}{2} + \frac{3}{4} \cdot 2 = 2$, whereas her net marginal valuation for product B is $1 - p_i^B \leq 2$. As $\beta > 1$, a best response allocates all positive consumption to a product with the highest net marginal valuation. Thus, consuming two units of product A is optimal.

At all other price profiles, select a global maximizer, over nonnegative consumption vectors, of the exact

potential

$$\Phi(\mathbf{x}^A, \mathbf{x}^B; \mathbf{p}^A, \mathbf{p}^B) = \sum_{s \in \{A,B\}} \left[(\mathbf{1} - \mathbf{p}^s)^\top \mathbf{x}^s - \frac{1}{2} (\mathbf{x}^s)^\top \left(I - \frac{3}{4} H \right) \mathbf{x}^s \right] - \beta (\mathbf{x}^A)^\top \mathbf{x}^B.$$

Such a maximizer exists because $I - \frac{3}{4}H$ is positive definite and Φ tends to $-\infty$ as consumption becomes unbounded on the nonnegative orthant. A unilateral change in a consumer's choices changes this potential by exactly the change in her payoff. Hence, every global maximizer is a consumer Nash equilibrium. The specified continuation strategies are therefore credible after every price history.

At the proposed equilibrium prices, $\mathbf{p}^{A*} = (\frac{1}{2}, \frac{1}{2})^\top$ and $\mathbf{p}^{B*} = \mathbf{0}$, these strategies yield $\mathbf{x}^{A*} = (2,2)^\top$ and $\mathbf{x}^{B*} = \mathbf{0}$. Firm A earns 2, whereas firm B earns zero. To verify firm A's incentives, consider any unilateral price deviation and the resulting consumer equilibrium. Since consumers consume only one product, every consumer with $x_i^A > 0$ satisfies $p_i^A = 1 - x_i^A + \frac{3}{4} x_j^A$ for $j \neq i$.

Multiplying by consumption and summing gives $\pi^A = \mathbf{1}^\top \mathbf{x}^A - (\mathbf{x}^A)^\top \left(I - \frac{3}{4} H \right) \mathbf{x}^A$. Inactive consumers contribute zero, so this identity also applies when the deviation changes the product consumed . Since $I - \frac{3}{4}H$ is positive definite, this expression is strictly concave in $\mathbf{x}^A$. Its first-order condition, we have $\mathbf{1} - 2\left(I - \frac{3}{4}H\right)\mathbf{x}^A = \mathbf{0}$, has the unique solution $\mathbf{x}^A = (2,2)^\top$, at which profit equals 2. Thus, $\pi^A \leq 2$ under any price deviation. Firm A already earns 2 at the proposed equilibrium and therefore cannot gain by deviating.

Inactive consumers contribute zero, so this identity also applies when the deviation changes the product consumed. Firm A therefore already attains its highest possible profit and cannot gain by deviating.

Next, consider any price deviation by firm B. If both prices remain at least -1, the specified continuation keeps both consumers consuming product A, and B continues to earn zero.

Otherwise, without loss of generality, suppose $p_1^B < -1$. With $\mathbf{p}^A = (\frac{1}{2}, \frac{1}{2})^\top$, the impossibility of consuming both products and consumers' optimal consumption choices imply

$$\begin{cases} x_1^A \leq \dfrac{1}{2} + \dfrac{3}{4} x_2^A, \\ x_2^A \leq \dfrac{1}{2} + \dfrac{3}{4} x_1^A, \end{cases} \qquad \Longrightarrow \qquad x_1^A, x_2^A \leq 2.$$

Since $x_2^B \geq 0$ and $p_1^B < -1$, we have $1 - p_1^B + \frac{3}{4} x_2^B > 2 \geq \frac{1}{2} + \frac{3}{4} x_2^A > 0$. Holding consumer 2's choices fixed,

consumer 1 therefore obtains strictly higher maximized utility from consuming product B alone:

$$\max_{x_1^B \geq 0} u_1(0, x_1^B) = \frac{1}{2}\left(1 - p_1^B + \frac{3}{4}x_2^B\right)^2 > \frac{1}{2}\left(\frac{1}{2} + \frac{3}{4}x_2^A\right)^2 = \max_{x_1^A \geq 0} u_1(x_1^A, 0) > 0.$$

Together with the impossibility of consuming both products when $\beta > 1$, this implies that consumer 1 must choose product B in any consumer equilibrium.

If only consumer 1 chooses product B, firm B earns $p_1^B(1 - p_1^B) < 0$. If both consumers choose product B, their consumption satisfies $\mathbf{x}^B = \left(I - \frac{3}{4}H\right)^{-1}\left[\mathbf{1} - \begin{pmatrix} p_1^B \\ p_2^B \end{pmatrix}\right]$. Substituting consumption into $\pi^B = p_1^B x_1^B + p_2^B x_2^B$ gives

$$\pi^B = \frac{16}{7}\left[\frac{7}{4}(p_1^B + p_2^B) - (p_1^B)^2 - (p_2^B)^2 - \frac{3}{2}p_1^B p_2^B\right] \leq 2 - \left(p_1^B - \frac{1}{2}\right)^2 < -\frac{1}{4} < 0$$

where the strict inequality follows from $p_1^B < -1$. Thus, even a deviation involving subsidies cannot generate positive profit for firm B.

Neither firm has a profitable deviation, and consumers play a Nash equilibrium after every price history. The first outcome is therefore supported by a subgame-perfect equilibrium. Exchanging the firm labels throughout the entire strategy profile, including consumers' continuation strategies, establishes the second equilibrium. □